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***Pure strategy equilibria of multidimensional  
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# PURE STRATEGY EQUILIBRIA OF MULTIDIMENSIONAL AND NON-MONOTONIC AUCTIONS

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**ABSTRACT.** We give necessary and sufficient conditions for the existence of symmetric equilibrium without ties in interdependent values auctions, with multidimensional independent types and no monotonic assumptions. In this case, non-monotonic equilibria might happen. When the necessary and sufficient conditions are not satisfied, there are ties with positive probability. In such case, we are still able to prove the existence of pure strategy equilibrium with an *all-pay auction tie-breaking rule*. As a direct implication of these results, we obtain a generalization of the Revenue Equivalence Theorem. From the robustness of equilibrium existence for all-pay auctions in multidimensional setting, an interpretation of our results can give a new justification to the use of tournaments in practice.

**JEL Classification Numbers:** C62, C72, D44, D82.

**Keywords:** auctions, pure strategy equilibria, non-monotonic bidding functions, tie-breaking rules

## 1. INTRODUCTION

The received literature on pure strategy equilibria on auctions is mainly restricted to the setting of unidimensional types and monotonic utilities. Although recent efforts have been made to treat the case of multidimensional types (see McAdams (2003), for instance), the monotonicity assumption is usually maintained. In dealing with multidimensional types, this is obviously restrictive (see also our examples in section 5).

Thus, to develop a satisfactory theory of equilibria with multidimensional types, it is necessary to take in account the possibility of non-monotonic utility functions.

However, even in the unidimensional case, non-monotonic auctions are problematic. To see why, consider a symmetric first-price auction between two buyers with payoff function  $v(t_i, t_{-i}) = \alpha + t_i + \beta t_{-i}$  and independent types distributed on  $[0, 1]$ .

The received theory ensures the existence of a monotonic *pure* strategy equilibrium only if  $\beta \geq 0$ . See Milgrom and Weber (1982), Maskin and Riley (2000), Athey (2001). If  $\beta < 0$ , we only know that there exists a tie-breaking rule (endogenously defined) that guarantees the existence of *mixed* strategy equilibrium (see Jackson, Simon, Swinkels and Zame (2002), henceforth JSSZ).

That the case  $\beta < 0$  is problematic can be seen through particular examples. Indeed, if  $\alpha = 5$ ,  $\beta = -4$  and the distribution is uniform on  $[0, 1]$ , this is exactly example 1 of JSSZ. If  $\alpha = 3$ ,  $\beta = -2$  and types assume values 0 or 1 with probabilities  $\frac{2}{3}$  and  $\frac{1}{3}$ , respectively, it is example 3 of Maskin and Riley (2000). Both cases are counterexamples to the existence of equilibrium, even with special tie-breaking rules. Maskin and Riley (2000) show that there is no equilibrium

for their example neither under the standard tie-breaking rule (that assigns the object randomly to tying bidders), nor under the Vickrey auction tie-breaking rule, defined as “if a tie occurs for the high bid, a Vickrey auction is conducted among the high bidders”. JSSZ make the claim, corrected in Jackson, Simon, Swinkels and Zame (2004), that there is no tie-breaking rule that is type-independent and ensures the existence of equilibrium for their example.

Some questions arise from the contrast between the theoretical results for  $\beta \geq 0$  and  $\beta < 0$ : For which set of  $\beta$  the standard tie-breaking rule is sufficient to ensure the existence of equilibrium? Is it possible to define a specific tie-breaking rule for all  $\beta$ ? For which set of  $\beta$  there is no equilibrium in pure strategy? Are the results valid only for first-price auctions? Is the equilibrium unique? Is the Revenue Equivalence Theorem still valid for  $\beta < 0$ ?

The framework of this paper includes JSSZ’s example as a special case. Our results provide the following answers to the above questions: If  $\beta > -1$ , there exists equilibrium in pure strategies under the standard tie-breaking rule. If we adopt an *all-pay auction tie-breaking rule*, that consists in conducting an all-pay auction among tying bidders in the case of a tie, then there exists a *pure* strategy equilibrium for all  $\beta$  (provided  $\alpha \geq \max\{0, -\beta\}$ , otherwise the object would have negative values). Moreover, the all-pay auction tie-breaking rule works for all standard type of auctions and the equilibria obtained under it obey the Revenue Equivalence Theorem. We also prove that there is a unique equilibrium if  $\beta > -1$ , but there are multiple equilibria otherwise.

It is important to note that the all-pay auction tie-breaking rule is *type-independent*, in the sense that it does not require private information. This does not contradict the example of Jackson et. al. (2004), that does not have equilibrium with type-independent tie-breaking rule. The reason is that the example is not a standard auction: there is an uncertainty about the number of objects in the auction.

Our results hold for symmetric auctions with independent non-atomic types, for a wide class of auction formats where bidders have unitary demands (first-price, second-price, all-pay, war of attrition).<sup>1</sup> Moreover, we impose no restriction on the dimension of the set of types and make no monotonic assumptions about the value of the object. All the answers provided above for the specific example are given in a general setting (of weakly separable utilities — see assumption H3 in section 5). Of course, the condition for the equilibrium existence is something more complex for the general case, but it is still easy to verify.

From the equilibrium existence for all-pay auctions, an interpretation of our results can give a new justification to the use of tournaments in practice. Indeed, tournaments (for job or research) are well-modeled as all-pay auctions or war of attrition. We prove that the existence of equilibrium for these kind of auctions require weaker assumptions than other kind of auction mechanisms, because they better reveal information. If we are in a situation where the revelation of information is crucial for a *strategically stable* allocation of the product, then our results can be read as saying that all-pay auctions are better. It is interesting to observe that the situations where tournaments are routinely conducted are exactly those where the information is multidimensional and can be non-monotonic for the players. For instance, the better capacity for conducting a research depends on a multidimensional vector of characteristic: technical

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<sup>1</sup>The non-atomicity is a standard but crucial assumption for our method. Thus, example 3 of Maskin and Riley (2000) does not satisfy our assumptions.

knowledge and abilities, experience, organizational and financial structures, creativity, motivation and even honesty. Such a complex information environment requires a better mechanism for revelation of information.

Our results are based in what we call *indirect auctions*, which we describe in subsection 1.1 below.

**1.1. The Indirect Auction.** For standard auctions, higher bids correspond to higher probability of winning. If a bidding function  $b(\cdot)$  is fixed and followed by all participants in a symmetric auction, we can associate to each bid (and thus, to each type), the probability of winning. All types that bid the same bid under  $b(\cdot)$  have the same probability of winning. This allows us to introduce the concept of conjugation. If  $b(t) = b(s)$ , and hence,  $t$  and  $s$  have the same probability of winning, we say that  $t$  and  $s$  are conjugated.

Sometimes in the literature, what we call conjugation is named *reduced form*: “The function relating a bidder’s type to his probability of winning is the reduced form of the auction.” (Border, 1991, p. 1175).<sup>2</sup> Therefore, what we will call *indirect auction* can be also called *reduced form auction*. The papers about reduced form auctions analyze problems related to the characterization and existence of optimal auctions. Hence, the auction is treated, as Myerson (1981) does, only by considering the probability of winning and the payments. In turn, our problem is to find the equilibrium for fixed auction rules. Moreover, our indirect auction is not equivalent to the direct one. Thus, it is not merely a reduced form of the auction. (See remarks after Theorem 1 in section 4). In the light of these differences and in the attempt not to confuse terms, we decided to use a different terminology.

This terminology comes from the Taxation Principle which allows us to implement any direct truthful mechanism through some convenient indirect one.<sup>3</sup> In this case, we are implementing the equilibrium in the auction using an indirect auction obtained from the reparametrization of types through the probability of winning.

This method allows us to deal with non-monotonic bidding in equilibrium. Indeed, we give examples where bidders’ types are multidimensional and the values are non-monotonic.<sup>4</sup> An important part of the method is the necessary condition (i) in Theorem 1, which says that the types of the bidder choosing the same bid have the same marginal benefit. A similar property was derived by Araujo and Moreira (2000) in the monopolistic principal-agent screening problem where non-monotonic optimal contracts emerge.

Returning to the description of our method, the main idea is to reparameterize types and to associate them to the probability of winning the auction. As stated, this idea seems unpromising at first, since the probability of winning will be different for each different bidding function that we begin with. Moreover, if we do not previously fix a bidding function, no conjugation can be defined.

To overcome these problems, we define conjugations without using bidding functions, as a suitable reparametrization of the types. Once we defined conjugations, we can define the Indirect Auction.<sup>5</sup> For this, we simply integrate the utilities of the direct auction for all types that are conjugated. From our definition of conjugation, the indirect auction is now an auction

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<sup>2</sup>See also Matthews (1984) and Chen (1986).

<sup>3</sup>See Guesnerie (1998).

<sup>4</sup>See sections 4 and 5.

<sup>5</sup>See section 3.

with the same format of the direct one (for instance, a first-price auction if the original auction is a first-price auction) between two players with independent signals, uniformly distributed on  $[0, 1]$ . This makes the analysis of equilibrium existence easier.

In section 2, we describe the model. Section 3 formally presents the indirect auction. Section 4 develops the theory for general auctions, obtaining necessary and sufficient conditions for the existence of equilibrium. Section 5 particularizes to the case of weakly separable utilities and gives a concise condition for equilibrium existence. Moreover, the all-pay auction tie-breaking rule is introduced and the equilibrium existence proved. As a corollary, we obtain the Revenue Equivalence Theorem. Section 6 concludes with a discussion about the limitations of our results and reviews the contributions of the paper in light of the related literature. All proofs are collected in appendices.

## 2. THE MODEL

There are  $N$  bidders in an auction of  $L < N$  homogenous objects, but each bidder is interested in just one object. Player  $i$  ( $i = 1, \dots, N$ ) receives a private information,  $t_i$ , possibly multidimensional, and chooses a bid  $b_i \in B \equiv \{b_{OUT}\} \cup [b_{\min}, +\infty)$ , where  $b_{\min} > b_{OUT}$  is the minimal valid bid and if  $b_i = b_{OUT}$ , bidder  $i$  does not participate in the auction and gets a payoff of 0.

Let  $t = (t_i, t_{-i})$  be the profile of all signals and  $b = (b_i, b_{-i})$ , the profile of submitted bids. Let  $b_{(m)}^{-i}$  be the  $m$ -th order statistic of  $(b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_N)$ , that is,  $b_{(1)}^{-i} \geq b_{(2)}^{-i} \geq \dots \geq b_{(N-1)}^{-i}$ . Since there are  $L$  objects, the value that determines the winning and losing events for bidder  $i$  is  $b_{(-i)} \equiv \max \{b_{\min}, b_{(L)}^{-i}\}$ . That is, the bidder  $i$  receives an object if  $b_i > b_{(-i)}$  and none if  $b_i < b_{(-i)}$ . Ties ( $b_i = b_{(-i)}$ ) are broken by the standard tie-breaking rule, that is, the object is randomly divided among the tying bidders. More specifically, the payoff of bidder  $i$  is given by

$$u_i(t, b) = \begin{cases} v(t_i, t_{-i}) - p^W(b_i, b_{(-i)}), & \text{if } b_i > b_{(-i)} \\ -p^L(b_i, b_{(-i)}), & \text{if } b_i < b_{(-i)} \\ \frac{v(t_i, t_{-i}) - b_i}{m}, & \text{if } b_i = b_{(-i)} \end{cases}$$

where  $v$  is the value of the object for all bidders,  $p^W$  and  $p^L$  are the payments made in the events of winning and losing, respectively, and  $m$  is the number of bidders tying.

Our setting is given by the following assumptions:

**(H0)** The types are independent and identically distributed in the same compact set  $S$ , according to a non-atomic probability measure  $\mu$  on  $S$ .  $v$  is positive ( $v > 0$ ), continuous and symmetric in its last  $N-1$  arguments, that is, if  $t'_{-i}$  is a permutation of  $t_{-i}$ ,  $v(t_i, t'_{-i}) = v(t_i, t_{-i})$ .

The restrictive aspect of (H0) is the symmetry. The others are very natural. For instance, the assumption that  $v$  is positive is not restrictive, since  $S$  is compact and, hence,  $v$  assumes a minimum  $M$ . Then, if we add  $M+1$  to the value of the object,  $v$  becomes positive.

The specific auction is determined by  $p^W$  and  $p^L$ . We will consider alternatively, two cases. The first one, embodied in (H1)-1 below, cover first-price auctions, for instance. The second case is defined by (H1)-2 and covers second price auctions, among other more exotic formats.

**(H1)** Over the domain  $B \times B$ ,  $p^W$  and  $p^L$  are non-negative, differentiable,  $p^L(b_{OUT}, \cdot) = 0$ ,  $\partial_1 p^W \geq 0$ ,  $\partial_1 p^L \geq 0$  and, alternatively:

- (H1)-1:  $\partial_1 p^W(\cdot) > 0$  or  $\partial_1 p^L(\cdot) > 0$  or  
(H1)-2:  $\partial_1 p^W = \partial_1 p^L \equiv 0$  and  $\partial_2(p^W - p^L) > 0$ .

Observe that assumption (H1) is rather weak. It covers virtually all kind of standard single-object auctions or multi-unit auctions with unitary demands, and allows the use of entry fees. Some examples are:

- (F) First-price auctions:  $p^W(b_i, b_{(-i)}) = b_i$  and  $p^L(b_i, b_{(-i)}) = 0$ .  
(S) Second-price auctions:  $p^W(b_i, b_{(-i)}) = b_{(-i)}$  and  $p^L(b_i, b_{(-i)}) = 0$ .  
(A) All-pay auctions:  $p^W(b_i, b_{(-i)}) = b_i$  and  $p^L(b_i, b_{(-i)}) = b_i$ .  
(W) War of attrition:  $p^W(b_i, b_{(-i)}) = b_{(-i)}$  and  $p^L(b_i, b_{(-i)}) = b_i$ .

An active reserve price, that is,  $b_{\min}$  that excludes some bidders, makes the statement of our equilibrium results more complex. So, we will postpone the analysis of this case to Appendix B and through the paper we will make use of the following assumption:

**(H2)**  $v$ ,  $p^W$ ,  $p^L$  and  $b_{\min}$  are such that no bidder plays  $b_{OUT}$ , that is, no bidder prefers to stay out of the auction.

We denote the auction described above by  $(S, \mu, v)$ . Observe that we make no restriction about the dimension of  $S$ . Also, we are considering just symmetric auctions. Thus, throughout the paper, when we talk about a strategy, we always mean a symmetric one. Under these assumptions we will introduce a new approach to prove existence of equilibria in auctions. We call it the “Indirect Auction Approach”. This is the subject of the next section.

### 3. THE INDIRECT AUCTION

In the subsection 3.1, we describe the basic element of our method: the conjugation. In subsection 3.2, the indirect auction is defined and its basic properties derived.

**3.1. Conjugations.** We will be interested in regular bidding functions as defined below:

**Definition 1.** A bounded measurable function  $b : S \rightarrow \mathbb{R}$  is regular if the c.d.f.

$$F_b(c) \equiv \Pr\{s \in S : b(s) < c\}$$

is absolutely continuous and strictly increasing in its support,  $[b_*, b^*]$ .

From the fact that  $F_b(\cdot)$  is absolutely continuous we conclude that  $F_b(c) = \Pr\{s \in S : b(s) \leq c\}$ . Let  $\mathcal{S}$  denote the set of regular functions. Observe that  $\mathcal{S}$  contains non-monotonic bidding functions. It is formed by functions  $b$  that do not induce ties with positive probability (because  $F_b$  is absolutely continuous) and that do not have gaps in the support of the bids (because  $F_b$  is increasing).

If a bidding function  $b \in \mathcal{S}$  is fixed, let us call the c.d.f. of the maximum bid of the opponents,  $\tilde{P}^b$ . That is, we define the transformation  $\tilde{P}^b : \mathbb{R}_+ \rightarrow [0, 1]$  by:

$$\begin{aligned} (1) \quad \tilde{P}^b(c) &= (\Pr\{t_i \in S : b(t_i) < c\})^{N-1} \\ &= \Pr\{t_{-i} \in S^{N-1} : b(t_j) < c, j \neq i\}. \end{aligned}$$

By the definition of  $\mathcal{S}$ ,  $\tilde{P}^b$  is strictly increasing and its image is the whole interval  $[0, 1]$ .

Now, we will denote by  $P^b : S \rightarrow [0, 1]$  the composition  $P^b = \tilde{P}^b \circ b$ . So, for a fixed  $b \in \mathcal{S}$ , followed by all players,  $P^b(t_i)$  is the probability of player  $i$  of type  $t_i$  wins the auction:

$$(2) \quad \begin{aligned} P^b(t_i) &= \Pr \{t_{-i} \in S^{N-1} : b_{(-i)}(t_{-i}) < b(t_i)\} \\ &= \Pr \{t_{-i} \in S^{N-1} : b(t_j) \leq b(t_i), \forall j \neq i\}. \end{aligned}$$

The following observation is important: from the symmetry required by (H0), the above function does not depend on  $i$  and  $P^b(t_i) \leq P^b(t_j)$  if and only if  $b(t_i) \leq b(t_j)$ . Obviously, two players have the same probability of winning if and only if they play the same bids. So, we have the following:

$$\{t_{-i} \in S^{N-1} : b_{(-i)}(t_{-i}) < b(t_i)\} = \{t_{-i} \in S^{N-1} : P_{(-i)}^b(t_{-i}) < P^b(t_i)\},$$

where  $P_{(-i)}^b(t_{-i}) \equiv \max_{j \neq i} P^b(t_j)$ . The equality of these events and (2) imply that

$$P^b(t_i) = \Pr \{t_{-i} \in S^{N-1} : P_{(-i)}^b(t_{-i}) < P^b(t_i)\}.$$

This observation will allow us to define conjugations without mentioning bidding functions. This will be very important in order to state our results. We have the following:

**Definition 2.** A conjugation for the auction  $(S, \mu, v)$  is a measurable and surjective function  $P : S \rightarrow [0, 1]$  such that for each  $i = 1, \dots, N$ ,

$$(3) \quad P(t_i) = \Pr \{t_{-i} \in S^{N-1} : P_{(-i)}(t_{-i}) \leq P(t_i)\} = [\Pr \{t_j \in S : P(t_j) < P(t_i), j \neq i\}]^{N-1}.$$

Observe that in the above definition, we do not need to mention the strategy  $b \in \mathcal{S}$ . It is also clear from the previous discussion that definition 2 is not empty, that is, for any regular function  $b \in \mathcal{S}$  there exists a conjugation defined by (2) that satisfies the above definition.

Observe also that, since the range of  $P$  is  $[0, 1]$ , we have, for all  $c \in [0, 1]$ ,

$$(4) \quad \Pr \{t_{-i} \in S^{N-1} : P_{(-i)}(t_{-i}) < c\} = c.$$

The above equation will be important in the sequel. It simply means that the distribution of  $P_{(-i)}(t_{-i})$  is uniform on  $[0, 1]$ .

Given  $b \in \mathcal{S}$ , equation (2) defines just one conjugation compatible with it. On the other hand, given a conjugation  $P$ , any function  $b \in \mathcal{S}$  that is an increasing transformation of  $P$  is compatible with  $P$ . To see this, suppose that there is an increasing function  $h : [0, 1] \rightarrow \mathbb{R}_+$ , such that  $b(t_i) = h(P(t_i))$  for  $\mu$ -almost all  $t_i \in S$ . Then,

$$\begin{aligned} P(t_i) &= \Pr \{t_{-i} : P(t_j) < P(t_i), \forall j \neq i\} \\ &= \Pr \{t_{-i} : h(P(t_j)) < h(P(t_i)), \forall j \neq i\} \\ &= \Pr \{t_{-i} : b(t_j) < b(t_i), \forall j \neq i\}. \end{aligned}$$

That is, given a conjugation  $P$ , there are many functions  $b \in \mathcal{S}$  compatible with it. In particular,  $b = P$  is a bidding function compatible with  $P$ .



**3.2. Indirect Auctions.** We proceed to define the indirect auction  $(\tilde{S}, \tilde{\mu}, \tilde{v})$  related to the direct auction  $(S, \mu, v)$ . The relation between them is given by the conjugation  $P : S \rightarrow [0, 1]$ . If the direct type of a player is  $t_i \in S$ , the indirect type will be  $P(t_i)$ . So,  $\tilde{S}$  is just  $[0, 1]$ . Each direct strategy  $b : S \rightarrow \mathbb{R}$  corresponds to an indirect strategy  $\tilde{b} : [0, 1] \rightarrow \mathbb{R}$ , such that the direct strategy will be the composition of the indirect strategy and the conjugation, that is,  $b = \tilde{b} \circ P$ , where  $P = P^b$ . What is this indirect strategy? Remember that  $P^b = \tilde{P}^b \circ b$  and  $\tilde{P}^b$  is increasing. So, given  $b$ , if we take the indirect strategy as  $\tilde{b} \equiv \left(\tilde{P}^b\right)^{-1}$ , then  $b = \tilde{b} \circ P$ , as we want. On the other hand, if it is given an indirect strategy  $\tilde{b}$  and a conjugation  $P$ , we have the associated direct strategy  $b = \tilde{b} \circ P$ . So we have just to define the indirect payoffs:

**Definition 3.** Fix a conjugation  $P$  for an auction  $(S, \mu, v)$ . The indirect utility function of bidder  $i$  associated to this conjugation is  $\tilde{v} : [0, 1]^2 \rightarrow \mathbb{R}$ , given by

$$(5) \quad \tilde{v}(x, y) \equiv E[v(t_i, t_{-i}) | P(t_i) = x, P_{(-i)}(t_{-i}) = y].$$

Now, fix a conjugation  $P$  and define the following function:

$$(6) \quad \tilde{\Pi}(x, c) \equiv E[\Pi(t_i, c) | P(t_i) = x],$$

where,  $\Pi(t_i, c)$  is the interim payoff of the direct auction. The notation should suggest to the reader that  $\tilde{\Pi}(x, c)$  will be the interim payoff of the indirect auction. Indeed, we have the following:

**Proposition 1.** Assume (H0). Given  $b \in \mathcal{S}$ , consider the corresponding conjugation  $P = P^b$  (defined by (2)) and the indirect bidding function  $\tilde{b} = \left(\tilde{P}^b\right)^{-1}$ . Alternatively, given a conjugation  $P$  and an indirect bidding function  $\tilde{b}$ , let  $b = \tilde{b} \circ P$  be the corresponding direct bidding function. Thus,

$$(7) \quad \tilde{\Pi}(x, c) = \int_0^{\tilde{b}^{-1}(c)} \left[ \tilde{v}(x, \alpha) - p^W(c, \tilde{b}(\alpha)) \right] d\alpha - \int_{\tilde{b}^{-1}(c)}^1 p^L(c, \tilde{b}(\alpha)) d\alpha.$$

(ii) Assume that  $P$  is such that for all  $s$  with  $P(s) = x$ , and for all  $x, y \in [0, 1]$ ,

$$(8) \quad \tilde{v}(x, y) = E[v(t) | P(t_i) = x, P_{(-i)}(t_{-i}) = y] = E[v(t) | t_i = s, P_{(-i)}(t_{-i}) = y].$$

Then, for all  $t_i$  such that  $P(t_i) = x$  and for all  $c \in B$ ,

$$(9) \quad \tilde{\Pi}(x, c) = \Pi(t_i, c).$$

**Proof.** See Appendix A. ■

Observe that, because of (7),  $\tilde{\Pi}(x, c)$  is formally equivalent to the interim payoff of an auction between two bidders, with signals uniformly distributed on  $[0, 1]$ , where the opponent is following the strategy  $\tilde{b}(\cdot)$  and the (common-value) utility function is given by  $\tilde{v}(x, \alpha)$ . So, we define the indirect auction as follows:

**Definition 4.** Given an auction  $(S, \mu, v)$  and a conjugation  $P$  for it, the associated indirect auction is an auction between two players with independent types uniformly distributed on  $[0, 1]$  and where the utility function is  $\tilde{v}$  defined by (5). The indirect auction is denoted by  $(\tilde{S}, \tilde{\mu}, \tilde{v})$  where  $\tilde{\mu}$  is the Lebesgue measure in  $\tilde{S} = [0, 1]$ .

The reader should keep in mind that the indirect auction is just an auxiliary and fictitious auction that will help in the analysis of the “direct” one. It is clear through definitions 1-4 how a conjugation relates the direct and the indirect auction. Obviously, a function  $\tilde{b} : [0, 1] \rightarrow \mathbb{R}_+$  is equilibrium of the indirect auction if for almost all  $x \in [0, 1]$ ,  $\tilde{\Pi}(x, \tilde{b}(x)) \geq \tilde{\Pi}(x, c)$ ,  $\forall c \in B = \{b_{OUT}\} \cup [b_{\min}, +\infty)$ .

#### 4. NECESSARY AND SUFFICIENT CONDITIONS FOR REGULAR EQUILIBRIA

The results and definitions of the two previous subsections allow us to show that the existence of a direct equilibrium implies the existence of the indirect one (Theorem 1, below). Conversely, (with an extra relatively weak assumption of consistency of payoffs) the existence of equilibrium in indirect auctions implies the existence in direct ones (Theorem 2).

**Theorem 1** (*Necessary Conditions*). Assume (H0)-(H2). If there is a pure strategy equilibrium  $b \in \mathcal{S}$  for the direct auction  $(S, \mu, v)$  and there exists  $\partial_b \Pi(s, b(s))$  for all  $s$ , then:

(i) the associated conjugation  $P = P^b$  (given by (2)) satisfies the following property: if  $s \in S$  is such that  $P(s) = x$ , then<sup>6</sup>

$$(10) \quad \tilde{v}(x, x) = E[v(t_i, t_{-i}) | P(t_i) = x, P_{(-i)}(t_{-i}) = x] = E[v(t_i, t_{-i}) | t_i = s, P_{(-i)}(t_{-i}) = x];$$

(ii) the indirect bidding function  $\tilde{b} = \left(\tilde{P}^b\right)^{-1}$ , where  $\tilde{P}^b$  is given by (1), is the increasing equilibrium of the indirect auction.

Moreover, if  $\tilde{v}$  is continuous (i.e., if it has a continuous representative), then:

(iii) (H1)-1 implies that  $\tilde{b}$  is differentiable and

$$(11) \quad \tilde{b}'(x) = \frac{\tilde{v}(x, x) - p^W(\tilde{b}(x), \tilde{b}(x)) + p^L(\tilde{b}(x), \tilde{b}(x))}{E_\alpha \left[ \partial_1 p^W(\tilde{b}(x), \tilde{b}(\alpha)) 1_{[\tilde{b}(x) > \tilde{b}(\alpha)]} + \partial_1 p^L(\tilde{b}(x), \tilde{b}(\alpha)) 1_{[\tilde{b}(x) < \tilde{b}(\alpha)]} \right]},$$

and (H1)-2 implies that

$$(12) \quad \tilde{v}(x, x) - p^W(\tilde{b}(x), \tilde{b}(x)) + p^L(\tilde{b}(x), \tilde{b}(x)) = 0;$$

(iv) the expected payment of a bidder with type  $t_i$  is given by

$$p(t_i) = \int_0^{P(t_i)} \tilde{v}(\alpha, \alpha) d\alpha;$$

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<sup>6</sup>This condition is related to an analogous one derived by Araujo and Moreira (2000) for the screening problem and Araujo and Moreira (2001) for signaling model. In these papers, the violation of the single crossing property leads to non-monotonicity.

(v) for all  $x$  and  $y \in [0, 1]$ ,

$$(13) \quad \int_y^x [\tilde{v}(x, \alpha) - \tilde{v}(\alpha, \alpha)] d\alpha \geq 0.$$

**Proof.** See Appendix C. ■

Theorem 1 says that if a multidimensional auction has a regular equilibrium, then it can be reduced to a unidimensional auction with two players (the indirect one). However, the reader should note that such reduction is non-trivial and that the indirect auction is not equivalent to the direct one. The indirect auction is a “fictitious” game, where each bidder is facing up a “fictitious” player, the “opponent”, that does not correspond to a real player. So, the dimension reduction is meant in this particular sense.

The expression in condition (iv) does not depend on the specific format of the payment rules,  $p^W$  and  $p^L$ , but it does depend on the conjugation. For the class of auctions considered in the next section, we are able to prove that the conjugation is unique and the Revenue Equivalence Theorem holds. On the other hand, condition (iv) plays an important role to prove the existence of equilibrium in the next result.

Theorem 2 is a kind of converse of Theorem 1. The main difference is that we do not require  $\tilde{v}$  to be continuous and we need condition (i)', which is slightly stronger than condition (i) in Theorem 1.

**Theorem 2** (*Sufficient Conditions*). Assume (H0)-(H2). Consider a direct auction  $(S, \mu, v)$ , a conjugation  $P$  and its associated indirect auction  $(\tilde{S}, \tilde{\mu}, \tilde{v})$ . Assume that

(i)' for all  $s \in S$  such that  $P(s) = x$ , and all  $y \in [0, 1]$ ,

$$(14) \quad \tilde{v}(x, y) = E[v(t_i, t_{-i}) | P(t_i) = x, P_{(-i)}(t_{-i}) = y] = E[v(t_i, t_{-i}) | t_i = s, P_{(-i)}(t_{-i}) = y];$$

(ii) for all  $x$  and  $y \in [0, 1]$ ,

$$\int_y^x [\tilde{v}(x, \alpha) - \tilde{v}(\alpha, \alpha)] d\alpha \geq 0;$$

(iii) there is an increasing function  $\tilde{b}$ , such that

$$\hat{p}(y) \equiv \int_0^y p^W(\tilde{b}(y), \tilde{b}(\alpha)) d\alpha + \int_y^1 p^L(\tilde{b}(y), \tilde{b}(\alpha)) d\alpha = \int_0^y \tilde{v}(\alpha, \alpha) d\alpha,$$

where  $\hat{p}(P(t_i)) = p(t_i)$  is the expected payment of a bidder of type  $t_i$ .

Then,  $\tilde{b}$  is the equilibrium of the indirect auction and  $b = \tilde{b} \circ P$  is the equilibrium of the direct auction. Moreover, if  $\tilde{v}$  is continuous, there exists  $\partial_b \Pi(s, b(s))$  for all  $s$  (which implies that all conditions of Theorem 1 are satisfied).

**Proof.** See Appendix C. ■

**Remark 1.** For the four specific formats, namely, the first-price auction (F), second-price auction (S), all-pay auction (A) and war of attrition (W), the function  $\tilde{b}$  is given, respectively, by

$$(15) \quad (F) \quad \tilde{b}(x) = \frac{1}{x} \int_0^x \tilde{v}(\alpha, \alpha) d\alpha$$

$$(16) \quad (S) \quad \tilde{b}(x) = \tilde{v}(x, x)$$

$$(17) \quad (A) \quad \tilde{b}(x) = \int_0^x \tilde{v}(\alpha, \alpha) d\alpha$$

$$(18) \quad (W) \quad \tilde{b}(x) = \int_0^x \frac{\tilde{v}(\alpha, \alpha)}{1 - \alpha} d\alpha$$

Conditions (iii) and (iv) reduce to the requirement that the function  $\tilde{b}$  above is increasing. In particular, the equilibrium may exist for an all-pay auction, for instance, but not for a first-price auction.

**Remark 2.** Although natural, condition (i)' can be still too restrictive. We need it in order to apply Proposition 1 and reach the conclusion that for all  $t_i$  such that  $P(t_i) = x$  and for all  $c \in R$ , we have:  $\tilde{\Pi}(x, c) = \Pi(t_i, c)$  (see (9) in Proposition 1). In turn, this implies that the equilibrium of the indirect auction is equilibrium of the direct auction. So, instead of assuming condition (i)' above, it would be sufficient to require the (necessary) condition (i) of Theorem 1 and that (9) is valid. For instance, when  $v(t_i, t_{-i})$  is given by  $\sum_{k=1}^K f_k(t_i) g_k(t_{-i})$ , condition (i) is sufficient to have (9) and Theorem 2 is valid with the (necessary) condition (i) in the place of the condition (i)'.<sup>7</sup>

Theorem 2 reduces the problem of equilibrium existence to find a conjugation that meets requirements (i)', (ii) and (iii). In the next section we treat a still general case (weakly separable auctions) where such conjugation can be easily defined. Nevertheless, we would like to give two examples where the assumptions of the next section are not satisfied. These examples illustrate a kind of heuristics for the existence problem. In example 1, we have a monotonic equilibrium and also a U-shaped one, which shows that the conjugation is not unique. In example 2, there is no monotonic equilibrium, but there is a bell-shaped equilibrium. Another example where Theorem 2 can be applied is an example provided by Jehiel, Meyer-ter-Vehn, Moldovanu and Zame (2004).

**Example 1** — Consider a symmetric first-price auction with two bidders, types uniformly distributed on  $[0, 1]$  and utility function given by

$$v(t_i, t_{-i}) = t_i + (3 - 4t_i + 2t_i^2) t_{-i}.$$

Observe that  $\partial_{t_i} v(t_i, t_{-i}) = 1 - 4t_{-i} + 4t_i t_{-i}$  can be negative. Thus, the received theory cannot be applied. Nevertheless, there exists a monotonic equilibrium. Indeed, in this case, the conjugation will be given by  $P(t_i) = t_i$  and we obtain

$$\tilde{v}(x, y) = x + (3 - 4x + 2x^2) y.$$

$\tilde{v}$  clearly satisfies condition (i)'. Condition (ii) follows from the fact that  $x > y$  implies

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<sup>7</sup>See also an extension of Example 1 in the Appendix D, which exemplifies the method in this case.

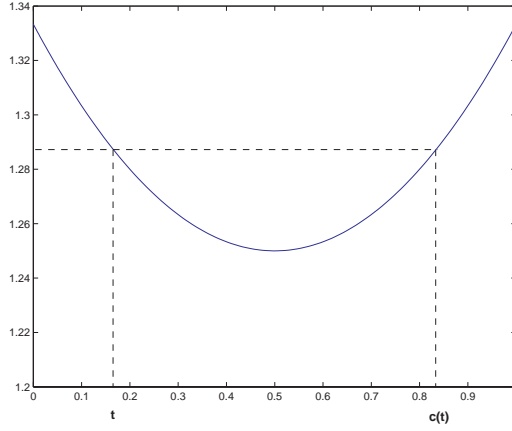


FIGURE 1. Equilibrium bidding function in Example 1.

$$\int_y^x [\tilde{v}(x, z) - \tilde{v}(z, z)] dz = \frac{(x - y)^2}{6} [3 + 3x^2 - 8y + 3y^2 + x(-4 + 6y)] \geq 0.$$

Condition (iii) is also satisfied, because the function

$$\tilde{b}(x) = \frac{1}{x} \int_0^x \tilde{v}(z, z) dz = \frac{x(24 - 16x + 3x^2)}{12}$$

is increasing. Clearly, the above function satisfies condition (iv). Thus, there exists a monotonic equilibrium by Theorem 2.

Nevertheless, this is not the unique equilibrium. If we assume that there exists a U-shaped equilibrium, the conjugation can be expressed by  $P(t_i) = |c(t_i) - t_i|$ , where  $c(t_i)$  is the type that bids as  $t_i$  (see Figure 1). Observe that  $c \circ c(t_i) = t_i$ . Condition (i) of Theorem 1 requires that

$$s + (3 - 4s + 2s^2) \frac{s + c(s)}{2} = c(s) + (3 - 4c(s) + 2c(s)^2) \frac{s + c(s)}{2},$$

that is,

$$s - c(s) = [s - c(s)] [4 - 2c(s) - 2s] \frac{s + c(s)}{2},$$

which simplifies to  $[s + c(s)] [2 - s - c(s)] = 1 \Rightarrow s + c(s) = 1$ . Then,  $c(s) = 1 - s$  and  $P(s) = |1 - 2s|$ . This gives the expression:

$$\tilde{v}(x, y) = \frac{5 + x^2}{4}$$

and condition (i)' and (ii) are easily seen to be satisfied. Also, condition (iii) and (iv) are satisfied, since

$$\tilde{b}(x) = \frac{1}{x} \int_0^x \tilde{v}(z, z) dz = \frac{5}{4} + \frac{x^2}{12}$$

is increasing. Then,  $b(s) = \frac{5}{4} + \frac{(1-2s)^2}{12}$  is a direct equilibrium, plotted in Figure 1.

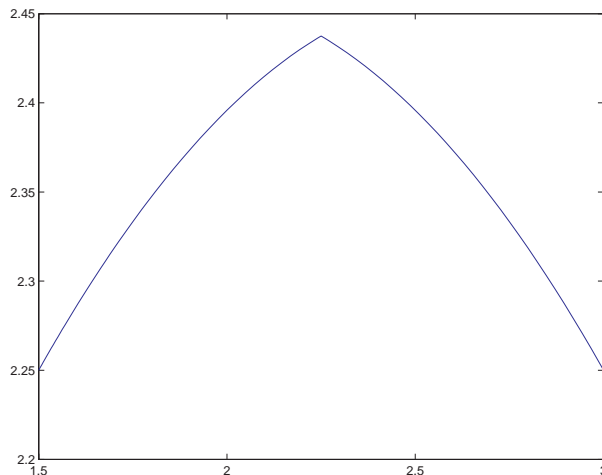


FIGURE 2. Equilibrium bidding function in Example 2.

Observe that no tie rules are needed in this case, because ties occur with zero probability. However, for each equilibrium bid, exactly two types pool and have the same probability of winning.

In Appendix D, we treat a slightly more general case, where the conjugation is more difficult to find than in this example. ■

**Example 2** — Consider again a symmetric first-price auction with two bidders and signals uniformly distributed in  $[1.5, 3]$  such that the value of the object is given by  $v(t_i, t_{-i}) = t_i(t_{-i} - \frac{t_i}{2})$ . In Appendix D, we show that this auction does not have monotonic regular equilibria, but there is a bell-shaped equilibrium as shown in Figure 2. ■

Example 1 shows that it is possible for a standard auction to have multiple equilibria. Example 2 suggests that the correct conjugation can fail to exist — at least with a fixed shape (that we began assuming). Thus, one would be interested in cases where it is possible to ensure the uniqueness of the equilibrium and where it is possible to find explicitly the conjugation. We do this under the context of assumption H3, to be presented in the next subsection.

## 5. EQUILIBRIUM EXISTENCE OF WEAKLY SEPARABLE AUCTIONS

Theorem 2 teaches us that the question of equilibrium existence is solved if we are able to find the proper conjugation. In examples 1 and 2 of the previous section we have shown situations where the conjugations could be obtained. However, there we assumed some features of the conjugation that are not necessary and we were able to find the correct conjugation for those settings. Now we will work in a setting where a conjugation always exists: the weakly separable auctions. These are the auctions satisfying the following assumption:

**(H3)** (Weak Separability).  $v(t_i, t_{-i})$  is such that if  $v(t_i, t_{-i}) < v(t'_i, t_{-i})$  for some  $t_{-i}$  then  $v(t_i, t'_{-i}) < v(t'_i, t'_{-i})$  for all  $t'_{-i}$ . Moreover, if  $C \subset \mathbb{R}$  has zero Lebesgue measure, then  $\mu\{s \in S :$

$v^1(s) \in C\} = 0$ , where

$$v^1(s) \equiv E[v(t_i, t_{-i}) | t_i = s]$$

is the expected value of the object for bidder with type  $s$ .

Assumption (H3) is restrictive, but it is valid in many economic meaningful cases.<sup>8</sup> Of course, private values are included in (H3).

Under (H3), we can define explicitly the conjugation:

$$(19) \quad P(t_i) \equiv \Pr\{t_{-i} \in S^{N-1} : v^1(t_j) < v^1(t_i), j \neq i\}.$$

Moreover, we can give a *necessary and sufficient* condition for the equilibrium existence of the direct auction: merely that the solution  $\tilde{b}$  to the first-order condition of the indirect auction be increasing. This is the content of the following:

**Theorem 3** (*Necessary and Sufficient Condition For Equilibrium Existence*). Assume (H0)-(H3). Let  $P$  be defined by (19) and let  $\tilde{v}$  be given by (5) for this  $P$ . There exists an equilibrium  $b \in \mathcal{S}$  if there exists an increasing function  $\tilde{b}$  that satisfies

$$(20) \quad \int_0^y p^W(\tilde{b}(y), \tilde{b}(\alpha)) d\alpha + \int_y^1 p^L(\tilde{b}(y), \tilde{b}(\alpha)) d\alpha = \int_0^y \tilde{v}(\alpha, \alpha) d\alpha.$$

If this is the case, the equilibrium of the direct auction is given by  $b = \tilde{b} \circ P$  and the expected payment of a bidder of type  $s$  is given by

$$(21) \quad p(s) = \int_0^{P(s)} \tilde{v}(\alpha, \alpha) d\alpha.$$

Additionally, if  $\tilde{v}$  is continuous, then there exists an equilibrium  $b \in \mathcal{S}$  and there exists  $\partial_b \Pi(s, b(s))$  for all  $s$  if and only if there exists an increasing function  $\tilde{b}$  that satisfies the following:

- For (H1)-1,  $\tilde{b}$  is differentiable and

$$\tilde{b}'(x) = \frac{\tilde{v}(x, x) - p^W(\tilde{b}(x), \tilde{b}(x)) + p^L(\tilde{b}(x), \tilde{b}(x))}{E_\alpha \left[ \partial_1 p^W(\tilde{b}(x), \tilde{b}(\alpha)) 1_{[\tilde{b}(x) > \tilde{b}(\alpha)]} + \partial_1 p^L(\tilde{b}(x), \tilde{b}(\alpha)) 1_{[\tilde{b}(x) < \tilde{b}(\alpha)]} \right]};$$

with initial condition  $\int_0^1 p^L(\tilde{b}(0), \tilde{b}(\alpha)) d\alpha = 0$ ;

- For (H1)-2,  $\tilde{b}$  is continuous and satisfies, for all  $x \in (0, 1)$ ,

$$\tilde{v}(x, x) - p^W(\tilde{b}(x), \tilde{b}(x)) + p^L(\tilde{b}(x), \tilde{b}(x)) = 0.$$

Moreover, if there is a unique  $\tilde{b}$  that satisfies such properties, the equilibrium of the direct auction in regular pure strategies is also unique.

**Proof.** See Appendix C. ■

<sup>8</sup>Theorem 3 of Debreu (1960) implies that if  $S$  is connected and  $v$  satisfies (H3), then  $v(t_i, t_{-i})$  can be written as  $h(u^1(t_i) + u^2(t_{-i}))$ , where  $h$  is an increasing function. In this case, (H3) would also imply that  $u^1(t_i)$  does not assume a value with positive probability.

**Remark 3.** As explained in Remark 1, if a multidimensional auction has a regular equilibrium, it can always be reduced (in a non-trivial way) to a one dimension auction (the indirect auction). So, for obtaining equilibrium existence, we have to consider auctions that can be “reduced”. This is what assumption H3 allows us to explicitly do. It still encompasses cases where such reduction is not trivial, as we show in examples 3 and 4 below. The reduction of the dimension of types is not a novelty in auction theory. While studying the efficiency of auctions, Dasgupta and Maskin (2000) use a condition close to H3 and Jehiel, Moldovanu and Stacchetti (1996) made such reduction for revenue maximization. Nevertheless, to show equilibrium existence in auctions, one cannot use only H3 or the Dasgupta and Maskin’s condition, since the received theory would require the monotonicity assumption of  $\tilde{v}$  on the reparameterized types. As we show in examples 4 and 5, this is not always possible. So, an important feature of Theorem 3 is that it does not require  $\tilde{v}$  to be monotonic.

**Example 3** (*Spectrum Auction*).<sup>9</sup>

Consider a first-price auction of a spectrum license. The license covers two periods of time:

(1) In the first period, the regulator lets the winner explore its monopoly power. Let  $t_i^1$  be the estimative of bidder  $i$  of the monopolist surplus in this first period. Of course, the true surplus will be better approximated by  $(t_1^1 + \dots + t_N^1)/N$ . If the bidder  $i$  (a firm) wins the auction, it has to invest  $t_i^2$ , a privately known amount, to build the network that will support the service. So, in the first period, the license gives to the firm

$$\frac{t_1^1 + \dots + t_N^1}{N} - t_i^2.$$

(2) In the second period, the regulator makes an estimate of the operational costs of the firm. The regulator cannot observe the true operational cost,  $t_i^3$ , which is a private information of the firm. Nevertheless, the regulator has a proxy that is a sufficient statistic for the mean operational cost of all participants in the auction,  $(t_1^3 + \dots + t_N^3)/N$ . The regulator will fix a price that will give zero profit for a firm with the mean operational costs.<sup>10</sup> So, in the second period, the license gives to the winner

$$\frac{t_1^3 + \dots + t_N^3}{N} - t_i^3.$$

So, the value of the object is given by

$$v(t_i, t_{-i}) = \frac{t_1^1 + \dots + t_N^1}{N} - t_i^2 + \frac{t_1^3 + \dots + t_N^3}{N} - t_i^3.$$

Let the signals  $t_i = (t_i^1, t_i^2, t_i^3)$ ,  $i = 1, \dots, N$ , be independent. Observe that the problem cannot be reduced to a single dimension. Indeed, if we summarize the private information by, say,  $s_i = t_i^1/N - t_i^2 + t_i^3(1/N - 1)$ , we lose the information about  $t_i^1$  and  $t_i^3$  that are needed for the value function of bidders  $j \neq i$ . Also, the model cannot be reparameterized to an increasing one. If we try to put  $-t_i^3$  in the place of  $t_i^3$ , then the dependence of  $v(t_i, t_{-i})$  on the signals  $t_j^3$  will be decreasing. So, the received theory does not ensure the existence of pure strategy equilibrium for this case. Nevertheless, assumption (H3) is trivially satisfied. In Appendix D, we assume

<sup>9</sup>This example is more complex, but formally similar to example 5 of Dasgupta and Maskin (2000).

<sup>10</sup>We assume that the regulator is institutionally constrained to follow such a procedure, so the optimality of this regulation is not an issue here.



the  $t_i = (t_i^1, t_i^2, t_i^3)$  are independent and uniformly distributed on  $[\underline{s}^1, \bar{s}^1] \times [\underline{s}^2, \bar{s}^2] \times [\underline{s}^3, \bar{s}^3]$ , with  $\underline{s}^1, \underline{s}^2, \underline{s}^3 \geq 0$  and we show that a sufficient condition for the existence of equilibrium in pure strategy is

$$\frac{\underline{s}^1}{N} - \bar{s}^2 - \bar{s}^3 \frac{N-1}{N} - 1 \geq 0.$$

The derivation in Appendix D indeed provides necessary and sufficient conditions for the existence of equilibrium. ■

**Example 4** (*Job Market*).

We model the job market for a manager as an auction among competing firms, where the object is the job contract. It is natural to assume that the manager has a multidimensional vector of characteristics,  $m = (m^1, \dots, m^K)$ . For the sake of simplicity, we assume that firms learn such characteristics through interviews and curriculum analysis. Each firm also has a position to be filled by the manager, with specific requirements for each dimension of the characteristics. For instance, if dimension 1 is ability to communicate and the position is to be the manager of a production section, there is level of desirability of this ability. An overly communicative person may not be good. The same goes for the other characteristics. A bank may desire a sufficiently (but not exaggeratedly) high level of risk loving or audacity on the part of the manager, while a family business may desire a much lower level. Even efficiency or qualification can have a level of desirability. Sometimes, the rejection of a candidate is explained by over-qualification. Therefore, let  $t_i = (t_i^1, \dots, t_i^K)$  be the value of the characteristics desired by the firm.

Since firms are competitors, then if one hires the employee, the other will remain with a vacant position, at least for a time.<sup>11</sup> In this way, the winning firm also benefits from the fact that its competitors have a vacant position — and, then, are not operating perfectly well. The higher the abilities required for the job, the more the competitor suffers.<sup>12</sup> So, the utility in this auction is as

$$v(t_i, t_{-i}) = \sum_{k=1}^K a^k m^k - \sum_{k=1}^K b^k (t_i^k - m^k)^2 + \sum_{j \neq i} \sum_{k=1}^K c^k t_j^k,$$

where  $a^k$  is the level of importance of characteristic  $k$  of the manager,  $b^k > 0$  represents how important is the distance from the desired level  $t_i^k$  of the characteristic  $k$ , and  $c^k$  is the weight of the benefit that firm  $i$  receives from the fact that the competitors are lacking  $\sum_{j \neq i} t_j^k$  of the ability  $k$ . As in the previous example, we cannot simplify this model to a unidimensional monotonic model. In Appendix D we analyze the case where there is just one dimension ( $K = 1$ ), 2 players ( $N = 2$ ) and types are uniformly distributed on  $[0, 1]$ ,  $b = b^1 \geq 0$ . We show that when

<sup>11</sup>This model works only for non-competitive job markets. In other words, the buyers (the contracting firms) have no access to a market with many homogenous employees to hire. This is implicit when we model it as an auction. So, this is the reason why a firm that does not contract the manager suffers — it is not possible to find a suitable substitute instantaneously.

<sup>12</sup>If firms act in an oligopolistic market, it is possible to justify such externality through the fact that the vacant position influences the quality of the product delivered by the firms and, hence, the equilibrium in this market.

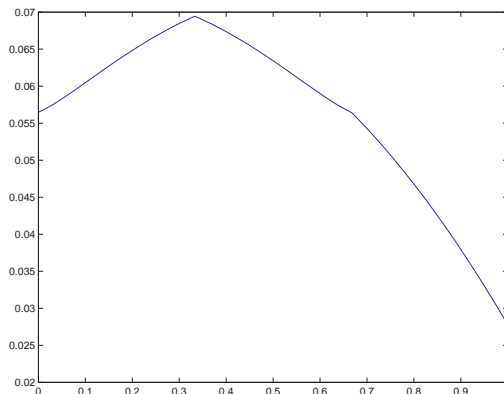


FIGURE 3. Equilibrium bidding function in Example 4.

$m^1 = m > 1/2$ , there exists a pure strategy equilibrium in regular strategies if and only if

$$c \equiv c^1 \geq \max \left\{ \frac{2b(m-2)}{3}, \frac{2b(1-2m)(1+m)}{3} \right\}$$

and when  $m < 1/2$ , if and only if

$$c \leq \min \left\{ \frac{2b(m+1)}{3}, \frac{2b(1-2m)(1+m)}{3} \right\}.$$

Observe that for both cases the value  $c = 0$  ensures the existence of equilibrium. This is expected, since it corresponds to a private value auction. For  $a = b = 1/5$ ,  $c = 1/20$  and  $m = 1/3$ , the equilibrium bidding function is shown in Figure 3. ■

Now, we can return to the example given in the introduction. Theorem 3 gives the conditions for the equilibrium existence.

**Example 5** (JSSZ, example 1). Let us consider a first price auction with two bidders, independent types uniformly distributed on  $[0, 1]$ . Let  $v^1(t_i) = t_i$  and  $v^2(t_{-i}) = \alpha + \beta t_{-i}$ . It is clear that  $P(t_i) = t_i$  in this case and  $\tilde{v}(x, y) = \alpha + x + \beta y$ . So,  $\tilde{v}(x, x) = \alpha + (1 + \beta)x$  and

$$\tilde{b}(x) = \frac{1}{x} \int_0^x \tilde{v}(z, z) dz = \frac{1}{x} \left[ \alpha x + \frac{(1 + \beta)x^2}{2} \right] = \alpha + \frac{(1 + \beta)x}{2},$$

which is increasing only if  $\beta > -1$ . Observe that for  $\tilde{b}(\cdot) \geq 0$ , it is necessary  $\alpha \geq -(1 + \beta)x/2$ , otherwise negative bids have to be allowed. ■

The example above is used by JSSZ to show that equilibrium may fail to exist under the standard tie-breaking rule. They then provide a general existence result based on endogenous tie-breaking rules. Nevertheless, their result has some undesirable properties. First, it is in mixed strategies. Second, the tie-breaking rule is endogenous, so it is not possible to know what rule has to be applied in order to guarantee the existence. Third, the rule requires that the players announce their types, which is theoretically convenient but unfeasible in the real world.

Instead, consider the following rule: if a tie occurs, conduct an all-pay auction among the tying bidders. If another tie occurs, split randomly the object.<sup>13,14</sup>

We show now that the all-pay auction tie-breaking rule ensures the existence of equilibrium for all auctions that we are considering.

**Theorem 4** (General Existence). Assume (H0) - (H3) and that the all-pay auction tie-breaking rule is adopted. If there is a continuous function  $\tilde{b}$ , not necessarily increasing, such that

$$\int_0^y p^W(\tilde{b}(y), \tilde{b}(\alpha)) d\alpha + \int_y^1 p^L(\tilde{b}(y), \tilde{b}(\alpha)) d\alpha = \int_0^y \tilde{v}(\alpha, \alpha) d\alpha,$$

then there exists a pure strategy equilibrium.

**Proof.** See Appendix C. ■

**Remark 4.** The main ingredients in the proof of Theorem 4 are the payment expression and the fact the bidding function of an all-pay auction is always increasing. This is so because in the all-pay auction, the bid

$$\tilde{b}(x) = \int_0^x \tilde{v}(\alpha, \alpha) d\alpha$$

is exactly the expect payment, which implies (20). Since  $v$  is positive by (H0),  $\tilde{v}$  is positive which implies that  $\tilde{b}$  is increasing. Thus, an auction that has an increasing  $\tilde{b}$ , as the war of attrition, can be also used as the tie-breaking mechanism.

**Example 5** (cont.) With the all-pay auction tie-breaking rule, the equilibrium of Example 5 is given, if  $\beta \leq -1$ , by the bid strategy  $b^1(x) = \bar{b}$ ,  $\forall x \in [0, 1]$ , in the first price auction, where  $\bar{b} \in \left[\alpha + \frac{1+\beta}{2}, \alpha\right]$ , and the bid strategy

$$b^2(x) = \alpha x + \frac{1+\beta}{2} x^2$$

for the tie-breaking (all-pay) auction. ■

**Remark 5.** When  $\tilde{b}$  is not increasing, there are types that are not ordered correctly (as in Example 5). This can be understood as a failure of  $\tilde{b}$  in correctly revealing the information that each bidder possesses. One can say that the tie-breaking rule has exactly the role of revealing information, in this sense. Thus, Theorem 4 can be interpreted as saying that all-pay auctions and war of attrition are better mechanisms for revealing information than first-price and second-price auctions. This can be another way of justifying the use of research tournaments in practice.<sup>15</sup> Indeed, the characteristics needed to compete for a research is very intricate: it is needed

<sup>13</sup>Observe that in the tie-breaking auction, bids and payments may be less than in the first auction.

<sup>14</sup>Maskin and Riley (2000) used a similar tie-breaking rule. They propose to conduct a second price (Vickrey) auction in the case of a tie.

<sup>15</sup>Research tournaments are theoretically modeled as all-pay auctions or war of attrition.

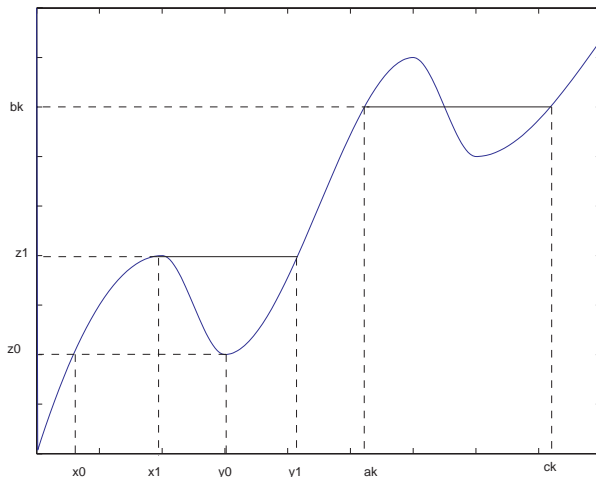


FIGURE 4. Possible specifications for the level of the tie.

information not only on the technical capabilities, but also on discipline, honesty, creativity, etc. This (multidimensional) information is better revealed through an all-pay auction.<sup>16,17</sup>

The reader should note that Theorem 4 does not claim the uniqueness of equilibrium. Indeed, if  $\tilde{b}$  is not increasing, there are many equilibria. There are two sources for this multiplicity.

The first source is that under the all-pay auction tie-breaking rule, any level of the bid in the range where  $\tilde{b}$  is not increasing can be chosen to be the level of the tie. This is shown in the Figure 4. For instance, any  $a_0$  can be chosen between  $x_0$  and  $x_1$ . Once one of the three elements  $a_k$ ,  $b_k$  or  $c_k$  is determined, so are the other two. However, these possibilities lead to the same expected payment and payoff for each bidder in the auction.

Another point is that the tie-breaking rule is not unique, in general. It can be shown, for instance, that for cases where  $\tilde{b}$  is decreasing (as in example 1 of JSSZ) and for some specifications of  $v$ , there is a continuum of tie-breaking rules (like that defined by JSSZ for their example 1), which ensures the existence of equilibrium. All these tie-breaking rules nevertheless imply different revenues. In light of this observation, the existence of equilibrium with an *endogenous* tie-breaking rule seems even more problematic as a solution concept, since it can sustain many different behaviors and payoffs at equilibrium.

This multiplicity is in contrast with the “ironing principle” usual in Contract Theory.<sup>18</sup> The unicity of the solution through the ironing principle comes from a Lagrangian condition that must be satisfied by the contract. Here, we do not have the maximization function of the principal. So, there is no additional condition that would fix the level of tying bids.

<sup>16</sup>Standard explanations for the use of tournaments in research also appeals to the role of information. See, for instance, Taylor (1995, p. 872): “Contracting for research is often infeasible because research inputs are unobservable and research outcomes cannot be verified by a court”. Our point is somewhat different, but obviously related to this explanation. The comparison among the information revealed by different auctions is the novelty.

<sup>17</sup>Obviously, these observations are valid only under the context of assumptions (H0)-(H3). It is a matter for future research to determine the range of validity of the existence ensured by all-pay auction tie-breaking rule.

<sup>18</sup>See, for instance, Guesnerie and Laffont (1984).

The reader may observe that the expression of the payment in Theorem 3 depends only on the conjugation, which is fixed for all kind of auctions. Also, the payment is exactly the same under the all-pay auction tie-breaking rule. So, we have the following:

**Theorem 5** (Revenue Equivalence Theorem). Consider auctions that satisfy (H0) -(H3) and with the all-pay auction tie-breaking rule. Then, any format of the auction gives the same revenue, provided that bidders follow the symmetric equilibrium specified by Theorem 4.

**Proof.** See Appendix C. ■

## 6. CONCLUSION

Now we will briefly highlight what are the most important contributions of this paper and discuss possible extensions.

**6.1. The Contributions.** Our contributions can be summarized as follows:

- **Equilibrium Existence in the Multidimensional Setting** — McAdams (2003a) generalizes Athey (2001) for multidimensional types and actions. He works with discrete bids and continuous types. Our approach gives the existence with continuum types and bids. Our result provides the expressions of the bidding functions, while his is an existence result only. His assumptions require monotonicity and are not on the fundamentals of the model. On the other hand, our results do not cover multi-unit auctions nor asymmetries as his. JSSZ give the existence for multidimensional games, including cases with dependence, while we require independence. However, they need an endogenous tie-breaking rule, and the existence is in mixed strategies, while our results are in pure strategies. Jackson and Swinkels (2004) show the existence of equilibria for a large class of multidimensional private value auctions. Their setting is private values, while ours is interdependent values. They allow asymmetries, dependence of signals and multi-units, but the existence is given in mixed strategies.
- **Equilibrium Existence in Non-Monotonic Settings** — We are not aware of any general non-monotonic equilibrium existence results in pure strategies. Zheng (2001), Athey and Levin (2001) and Ewerhart and Fieseler (2003) present cases where non-monotonicity arises.

Thus, our method develops a theory to deal with the situations where the usual monotonicity is not fulfilled.<sup>19</sup> Araujo and Moreira (2000) use a similar method for the screening problem without the single crossing property and Araujo and Moreira (2001) extend it to signaling model.

- **Uniqueness of Equilibrium** — We are able to ensure the uniqueness of equilibrium in the symmetric interdependent values auctions that satisfy assumption H3, extending the well known uniqueness of unidimensional and monotonic auctions.

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<sup>19</sup>Of course, papers that provide existence in distributional (mixed) strategies can treat non-monotonic settings as well.

- **Necessary and Sufficient Conditions for the Existence of Equilibrium without Ties** — The results of JSSZ do not allow one to distinguish when special tie-breaking is needed or when it is not. Our approach clarifies, under assumption H3, whether ties occur with positive probability (and there is a potential need for special tie-breaking rules).
- **All-Pay Auction Tie-Breaking Rule** — When there is a need for a tie with positive probability, we are able to offer an exogenous tie-breaking rule, which is implemented through an all-pay auction. Moreover, the equilibrium that the rule implements is in pure strategies. For private value auctions, Jackson and Swinkels (2004) show that the equilibrium is invariant for any trade-maximizing tie-breaking rule. Nevertheless, this does not need to hold for the interdependent values auctions that we treat here.
- **Information Revelation** — In the sense made precise by Remark 5, we show that all-pay auction and war of attrition are better mechanisms to reveal information than first-price and second-price auctions.
- **Revenue Equivalence Theorem** — We have also generalized the Revenue Equivalence Theorem (Theorem 5). Furthermore, Theorem 2 and Appendix B show that there is a deep connection between the revenue equivalence and the existence of equilibrium. Riley and Samuelson (1981) and Myerson (1981) establish that revenue equivalence is a consequence of the equilibrium behavior. Proposition 5 and Corollary 1 in Appendix B show that the revenue equivalence is also sufficient for the existence of equilibrium (if an extra condition is satisfied).

Thus, our results have clarified some aspects of the equilibrium existence problem in auctions. The theory shows that, under assumption H3, there is no additional difficulty in working with the more general setting of multidimensional types and non-monotonic utilities besides those difficulties already present in the unidimensional setting.<sup>20</sup> Moreover, this approach allows the equilibrium bidding functions to be expressed in a simple manner. This is so because the equilibrium bidding function of a general auction can be expressed by the equilibrium bidding function of an auction with two bidders and types uniformly distributed on  $[0, 1]$ .

**6.2. Limitations of the Method.** Our theory makes two important assumptions: independence of the types and symmetry.

The generalization of this approach for dependent types involves some difficulties, because the conjugation would depend in a complicated way on types. Nevertheless, we believe that some extension can be done if we assume conditional independence.<sup>21</sup> It is worth remembering that the problem with dependence is not specific to our approach. Jackson (1999) gives a counterexample for the equilibrium existence of an auction with bidimensional affiliated types. Fang and Morris (2003) also obtain negative results, not only for the existence of equilibrium but also for the revenue equivalence.

On the other hand, asymmetry does not seem to impose severe restriction on the existence of equilibrium. We believe that the approach of the indirect auction can be adapted to this case,

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<sup>20</sup>Theorem 3 shows that the non-existence of the equilibrium comes from the non-monotonicity of the indirect bidding function. This can also occur in an unidimensional setting, although it can be more usual in multidimensional models.

<sup>21</sup>de Castro (2004b) proposes the use of conditional independence as an alternative for affiliation.

although not in a straightforward way. If this can be done, it is unlikely that we will obtain simple expressions as in this paper.

Another limitation of our theory is that it is applied only to single-unit auctions. The risk neutrality does not seem to be a fundamental assumption, although complications can arise in extending the approach for risk aversion.

Finally, the relaxation of assumption H3 is an obvious direction to pursue, although H3 seems to encompass many important economical examples.

## Appendix A - Proof of the Basic Results

We will need the following result, which was proved, in a more general setting, by de Castro (2004a).

**Lemma 1 (Payoff Characterization)** — Assume (H0)-(H2). Fix  $b(\cdot) \in \mathcal{S}$ . The bidder  $i$ 's payoff can be expressed by

$$\Pi(t_i, b_i, b(\cdot)) = \Pi_i(t_i, b_{\min}) + \int_{(b_{\min}, b_i)} \partial_{b_i} \Pi(t_i, \beta, b(\cdot)) d\beta,$$

where  $\partial_{b_i} \Pi(t_i, \beta, b(\cdot))$  exists for almost all  $\beta$  with

$$\begin{aligned} \partial_{b_i} \Pi(t_i, \beta, b(\cdot)) = E \Big[ & -\partial_1 p^W(\beta, b_{(-i)}(t_{-i})) 1_{[\beta > b_{(-i)}(t_{-i})]} - \partial_1 p^L(\beta, b_{(-i)}(t_{-i})) 1_{[\beta < b_{(-i)}(t_{-i})]} \Big] \\ & + E[v(t_i, t_{-i}) - p^W(\beta, \beta) + p^L(\beta, \beta) | b_{(-i)}(t_{-i}) = \beta] f_{b_{(-i)}}(\beta). \end{aligned}$$

a.e., where  $b_{(-i)}(t_{-i}) \equiv \max_{j \neq i} b(t_j)$ .

**Proof.** Let us define  $b_{(-i)}(t_{-i}) \equiv \max_{j \neq i} b(t_j)$ . Since  $b_i = b_{(-i)}(t_{-i})$  with probability zero, we have

$$\begin{aligned} \Pi(t_i, b_i, b(\cdot)) &= \int_{S^{N-1}} [v(t_i, t_{-i}) - p^W(b_i, b_{(-i)}(t_{-i}))] 1_{[b_i > b_{(-i)}(t_{-i})]} \prod_{j \neq i} \mu(dt_j) \\ &+ \int_{S^{N-1}} [-p^L(b_i, b_{(-i)}(t_{-i}))] 1_{[b_i < b_{(-i)}(t_{-i})]} \prod_{j \neq i} \mu(dt_j). \end{aligned}$$

Take a sequence  $a_n \rightarrow b_i^-$ , i.e.,  $a_n < b_i$  (the other case is analogous). We want to prove that there exists  $\lim_{n \rightarrow \infty} D_n(b_i) / (b_i - a_n)$  for almost all  $b_i$ , where

$$D_n(b_i) = \Pi(t_i, b_i, b(\cdot)) - \Pi(t_i, a_n, b(\cdot)).$$

In the sequel, we will omit the measure  $\prod_{j \neq i} \mu(dt_j)$  and the terms  $t_{-i}$ . We have:

$$\begin{aligned}
D_n(b_i) &= \int [v(t_i, \cdot) - p^W(b_i, b_{(-i)}(\cdot))] 1_{[b_i > b_{(-i)}(\cdot)]} 1_{[a_n > b_{(-i)}(\cdot)]} \\
&\quad + \int [-p^L(b_i, b_{(-i)}(\cdot))] 1_{[b_i < b_{(-i)}(\cdot)]} \\
&\quad - \int [v(t_i, \cdot) - p^W(a_n, b_{(-i)}(\cdot))] \\
&\quad - \int [-p^L(a_n, b_{(-i)}(\cdot))] 1_{[a_n < b_{(-i)}(\cdot)]} \\
&= \int [v(t_i, \cdot) - p^W(b_i, b_{(-i)}(\cdot)) + p^L(a_n, b_{(-i)}(\cdot))] 1_{[b_i > b_{(-i)}(\cdot) > a_n]} \\
&\quad + \int [-p^W(b_i, b_{(-i)}(\cdot)) + p^W(a_n, b_{(-i)}(\cdot))] 1_{[a_n > b_{(-i)}(\cdot)]} \\
&\quad + \int [-p^L(b_i, b_{(-i)}(\cdot)) + p^L(a_n, b_{(-i)}(\cdot))] 1_{[b_i < b_{(-i)}(\cdot)]}
\end{aligned}$$

Let us call the three last integrals as  $D_n^1(b_i)$ ,  $D_n^2(b_i)$  and  $D_n^3(b_i)$ , respectively. Since  $p^W$  and  $p^L$  are differentiable, we have

$$\begin{aligned}
\lim_{n \rightarrow \infty} \frac{D_n^2(b_i)}{b_i - a_n} &= - \lim_{n \rightarrow \infty} \int \frac{p^W(a_n, b_{(-i)}(\cdot)) - p^W(b_i, b_{(-i)}(\cdot))}{b_i - a_n} 1_{[b_i > b_{(-i)}(\cdot)]} \\
&= - \int \partial_1 p^W(b_i, b_{(-i)}(\cdot)) 1_{[b_i > b_{(-i)}(\cdot)]}
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n \rightarrow \infty} \frac{D_n^3(b_i)}{b_i - a_n} &= - \lim_{n \rightarrow \infty} \int \frac{p^L(a_n, b_{(-i)}(\cdot)) - p^L(b_i, b_{(-i)}(\cdot))}{b_i - a_n} 1_{[a_n < b_{(-i)}(\cdot)]} \\
&= - \int \partial_1 p^L(b_i, b_{(-i)}(\cdot)) 1_{[b_i < b_{(-i)}(\cdot)]}.
\end{aligned}$$

everywhere. So, if we put  $a_0 = b_{\min}$  and  $b_i \geq b_{\min}$ , the Fundamental Theorem of Calculus gives

$$(22) \quad D_0^2(b_i) = \int_{(b_{\min}, b_i)} \int -\partial_1 p^W(\alpha, b_{(-i)}(\cdot)) 1_{[b_i > b_{(-i)}(\cdot)]} d\alpha$$

and

$$(23) \quad D_0^3(b_i) = \int_{(b_{\min}, b_i)} \int -\partial_1 p^L(\alpha, b_{(-i)}(\cdot)) 1_{[b_i < b_{(-i)}(\cdot)]} d\alpha.$$

Now, define the measure  $\rho$  over  $\mathbb{R}_+$  by

$$\rho(V) \equiv \int_{S^{N-1}} [v(t_i, t_{-i}) - p^W(b_i, b_{(-i)}(t_{-i})) + p^L(b_i, b_{(-i)}(t_{-i}))] 1_{[b_{(-i)}(t_{-i}) \in V]} \prod_{j \neq i} \mu(dt_j).$$

Observe that, since  $b \in \mathcal{S}$ ,  $\rho$  is absolutely continuous with respect the Lebesgue measure  $\lambda$ . We have

$$\lim_{n \rightarrow \infty} \frac{D_n^1(b_i)}{b_i - a_n} = \lim_{n \rightarrow \infty} \frac{\rho([a_n, b_i])}{b_i - a_n} = \lim_{a^n \rightarrow b_i} \left\{ \frac{\rho([a_n, b_i])}{\lambda([a_n, b_i])} \right\} = \frac{d\rho}{d\lambda}(b_i),$$



where  $\frac{d\rho}{d\lambda}(\cdot)$  is the Radon-Nikodym derivative of  $\rho$  with respect to  $\lambda$ . Indeed, the existence of such limit is ensured by Theorem 8.6 of Rudin (1966) for almost all  $b_i$ , that is,

$$\lambda \left( \left\{ b_i : \nexists \lim_{a^n \rightarrow b_i} \left\{ \frac{\rho([a_n, b_i])}{\lambda([a_n, b_i])} \right\} \right\} \right) = 0.$$

It is easy to see that the Radon-Nikodym derivative  $\frac{d\rho}{d\lambda}(b_i)$  is simply

$$E[v(t_i, t_{-i}) - p^W(\beta, \beta) + p^L(\beta, \beta) | b_{(-i)}(t_{-i}) = \beta] f_{b_{(-i)}}(\beta),$$

where  $f_{b_{(-i)}}(\beta)$  is the Radon-Nikodym derivative of the distribution of maximum bids,  $\int 1_{[b_{(-i)}(t_{-i}) \in V]}$ . Moreover, Theorem 8.6 of Rudin says that

$$(24) \quad \rho((b_{\min}, b_i)) = \int_{(b_{\min}, b_i)} \frac{d\rho}{d\lambda}(\alpha) d\alpha.$$

Thus, (22), (23) and (24) imply that

$$\Pi(t_i, b_i, b(\cdot)) = \Pi(t_i, b_{\min}) + \int_{(b_{\min}, b_i)} \partial_{b_i} \Pi(t_i, \beta, b(\cdot)) d\beta,$$

where

$$\begin{aligned} \partial_{b_i} \Pi(t_i, \beta, b(\cdot)) = E \Big[ & -\partial_1 p^W(\beta, b_{(-i)}(t_{-i})) 1_{[\beta > b_{(-i)}(t_{-i})]} - \partial_1 p^L(\beta, b_{(-i)}(t_{-i})) 1_{[\beta < b_{(-i)}(t_{-i})]} \\ & + E[v(t_i, t_{-i}) - p^W(\beta, \beta) + p^L(\beta, \beta) | \max_{j \neq i} b(t_j) = \beta] f_{b_{(-i)}}(\beta) \Big]. \end{aligned}$$

This concludes the proof. ■

**Proof of Proposition 1.** Let us introduce the following notation:

$$\begin{aligned} \Pi^+(t_i, c) &= \int [v(t_i, \cdot) - p^W(c, b_{(-i)}(\cdot))] 1_{[c > b_{(-i)}(\cdot)]} \Pi_{j \neq i} \mu(dt_j) \\ \Pi^-(t_i, c) &= \int p^L(c, b_{(-i)}(\cdot)) 1_{[c < b_{(-i)}(\cdot)]} \Pi_{j \neq i} \mu(dt_j), \\ \tilde{\Pi}^{+, -}(\phi_i, c) &\equiv E \left[ \Pi_i^{+, -}(t_i, c) | P(t_i) = \phi_i \right]. \end{aligned}$$

Let us begin with the proof for  $\tilde{\Pi}_i^+$  and  $\Pi_i^+$ . Let us denote the conditional expectation by

$$(25) \quad g^{t_i, c}(\alpha) \equiv E \left[ v(t_i, t_{-i}) - p^W(c, b_{(-i)}(t_{-i})) | P_{(-i)}^b(t_{-i}) = \alpha \right].$$

The event  $[c > b_{(-i)}(t_{-i})]$  occurs if and only if  $[\tilde{P}^b(c) > P_{(-i)}^b(t_{-i})]$  occurs. Then, we have

$$\Pi^+(t_i, c) = \int g^{t_i, c}(P_{(-i)}^b(t_{-i})) 1_{\tilde{P}^b(c) > P_{(-i)}^b(t_{-i})} \Pi_{j \neq i} \mu(dt_j).$$

Now we appeal to Lemma 2.2, p. 43, of Lehmann (1959). This lemma says the following: if  $R$  is a transformation and if  $\mu^*(B) = \mu(R^{-1}(B))$ , then

$$\int_{R^{-1}(B)} g[R(t)] \mu(dt) = \int_B g(\alpha) \mu^*(d\alpha).$$

In our case,  $R = P_{(-i)}^b$  and  $\mu^*([0, c]) = \mu^*([0, c)) = \tau_{-i} \left( (P^b)_{(-i)}^{-1}([0, c)) \right) = \Pr\{t_{-i} \in S^{N-1} : P^b(t_j) < c\} = c$ , by (4). So,  $\mu^*$  is exactly the Lebesgue measure, and we have

$$(26) \quad \Pi^+(t_i, c) = \int_0^{\tilde{P}^b(c)} g^{t_i, c}(\alpha) d\alpha.$$

From this and the definition of  $\tilde{\Pi}^+$ , we have

$$\begin{aligned} \tilde{\Pi}^+(\phi_i, c) &= E \left[ \int_0^{\tilde{P}^b(c)} g^{t_i, c}(\alpha) d\alpha \mid P^b(t_i) = \phi_i \right] \\ &= \int_0^{\tilde{P}^b(c)} E \left[ g^{t_i, c}(\alpha) \mid P^b(t_i) = \phi_i \right] d\alpha \\ &= \int_0^{\tilde{P}^b(c)} \left[ \tilde{v}(\phi_i, \alpha) - p^W(c, \tilde{b}(\alpha)) \right] d\alpha, \end{aligned}$$

where the second line comes from a interchange of integrals (Fubini's Theorem) and the last line comes from independency and the definition of  $\tilde{v}(\phi_i, \alpha)$  and  $g^{t_i, c}(\alpha)$  (see (5) and (25)). Also from the fact that  $\tilde{b} = (\tilde{P}^b)^{-1}$ , we can substitute  $\tilde{P}^b$  to obtain

$$(27) \quad \tilde{\Pi}^+(\phi_i, c) = \int_0^{\tilde{b}^{-1}(c)} \left[ \tilde{v}(\phi_i, \alpha) - p^W(c, \tilde{b}(\alpha)) \right] d\alpha.$$

Now, we can repeat the above procedures with  $\Pi^-(\phi_i, c)$  and obtain:

$$(28) \quad \tilde{\Pi}^-(\phi_i, c) = \int_{\tilde{b}^{-1}(c)}^1 p^L(c, \tilde{b}(\alpha)) d\alpha.$$

Adding up, that is, putting  $\tilde{\Pi}(\phi_i, c) = \tilde{\Pi}^+(\phi_i, c) - \tilde{\Pi}^-(\phi_i, c)$ , we obtain the interim payoff of the indirect auction. This concludes the proof of the first part.

For the second part, observe that the equality (8) implies that for all  $t_i$  such that  $P^b(t_i) = P^b(s) = x$ ,

$$\begin{aligned} E \left[ g^{t_i, c}(\alpha) \mid P^b(t_i) = x \right] &= E \left[ (v(t_i, t_{-i}) - p^W(c, b_{(-i)}(t_{-i}))) \mid P^b(t_i) = x, P_{(-i)}^b(t_{-i}) = \alpha \right] \\ &= E \left[ (v(t_i, t_{-i}) - p^W(c, b_{(-i)}(t_{-i}))) \mid t_i = s, P_{(-i)}^b(t_{-i}) = \alpha \right] \\ &= E \left[ (v(s, t_{-i}) - p^W(c, b_{(-i)}(t_{-i}))) \mid P_{(-i)}^b(t_{-i}) = \alpha \right] \\ &= g^{s, c}(\alpha). \end{aligned}$$

Then,

$$\begin{aligned}
\tilde{\Pi}^+(x, c) &= E \left[ \int_0^{\tilde{P}(c)} g^{t_i, c}(\alpha) d\alpha \mid P^b(t_i) = x \right] \\
&= \int_0^{\tilde{P}(c)} E \left[ g^{t_i, c}(\alpha) \mid P^b(t_i) = x \right] d\alpha \\
&= \int_0^{\tilde{P}(c)} g^{s, c}(\alpha) d\alpha \\
&= \Pi^+(s, c),
\end{aligned}$$

where the last line comes from (26). Obviously, the same can be done for  $\Pi^-$  and  $\tilde{\Pi}^-$ . So, the proof is complete. ■

## Appendix B - Indirect Auction Equilibria

In this appendix, we will analyze auctions between two players, with independent types uniformly distributed on  $[0, 1]$ . Since this is the setting of the indirect auction, we will use notation consistent with that, although the results of this appendix are independent from the results of section 4. For  $(i, -i) = (1, 2)$  or  $(2, 1)$  let

$$\tilde{u}_i(x, b) = \begin{cases} \tilde{v}(x_i, x_{-i}) - p^W(b_i, b_{-i}), & \text{if } b_i > b_{-i} \\ -p^L(b_i, b_{-i}), & \text{if } b_i < b_{-i} \\ \frac{\tilde{v}(x_i, x_{-i}) - b_i}{2}, & \text{if } b_i = b_{-i} \end{cases}$$

be the ex-post payoff. We will assume:

**(H0)** The types are independent and uniformly distributed on  $[0, 1]$ .  $\tilde{v}$  is positive, measurable and bounded above.

By the definition of the indirect auction, we are interested only in non-decreasing equilibria  $\tilde{b}$ , strictly increasing in the range of winning types. That is, for a non-decreasing strategy  $\tilde{b}$ , define  $x_0$  to be the minimum type that bids at least  $b_{\min}$ . So we require  $\tilde{b}$  to be increasing in  $[x_0, 1]$  and equal to  $\tilde{b}(x) = -1$  for  $x < x_0$ . In order to be an equilibrium,  $\tilde{b}$  must satisfy the following:

$$(29) \quad \int_0^{x_0} [\tilde{v}(x_0, \alpha) - p^W(\tilde{b}(x_0), \tilde{b}(\alpha))] d\alpha - \int_{x_0}^1 p^L(\tilde{b}(x_0), \tilde{b}(\alpha)) d\alpha = 0.$$

Indeed, the above integral is the payoff of  $x_0$ . If it is negative, then  $x_0$  can do better by bidding  $-1$ . If the above integral is positive, then  $x_0 > 0$ , otherwise the payoff of  $x_0 = 0$  would reduce to  $-\int_{x_0}^1 p^L(\tilde{b}(x_0), \tilde{b}(\alpha)) d\alpha$  which is non-positive because  $p^L$  is positive. If  $x_0 > 0$ , for a  $x < x_0$  sufficiently close of  $x_0$ , the payoff

$$\int_0^{x_0} [\tilde{v}(x, \alpha) - p^W(\tilde{b}(x_0), \tilde{b}(\alpha))] d\alpha - \int_{x_0}^1 p^L(\tilde{b}(x_0), \tilde{b}(\alpha)) d\alpha$$

is still positive (because of the continuity of  $\tilde{v}$ ). So,  $x$  is receiving zero but could obtain a strictly positive payoff by bidding  $\tilde{b}(x_0) = b_{\min}$ .

So, for a fixed  $\tilde{b}$ , we assume the following:

(H2)' There exists  $x_0$ , the minimum indirect type that satisfies (29) and  $x_0 \in [0, 1)$ .

We have the following:

**Proposition 2** (*Case 1*). Assume (H0)', (H1)-1, that is,  $\partial_1 p^W(\cdot) > 0$  or  $\partial_1 p^L(\cdot) > 0$ , (H2)', and that  $\tilde{v}$  is continuous. Let  $\tilde{b}$  be an increasing equilibrium of the indirect auction. Then,  $\tilde{b}$  is differentiable and satisfies

$$(30) \quad \tilde{b}'(x) = \frac{\tilde{v}(x, x) - p^W(\tilde{b}(x), \tilde{b}(x)) + p^L(\tilde{b}(x), \tilde{b}(x))}{E_\alpha \left[ \partial_1 p^W(\tilde{b}(x), \tilde{b}(\alpha)) 1_{[\tilde{b}(x) > \tilde{b}(\alpha)]} + \partial_1 p^L(\tilde{b}(x), \tilde{b}(\alpha)) 1_{[\tilde{b}(x) < \tilde{b}(\alpha)]} \right]}.$$

**Proof.** The proof is an adaptation of part of the proof of Theorem 2 of Maskin and Riley (1984).<sup>22</sup> Suppose that player  $-i$  follows  $\tilde{b}$ . The interim payoff of player  $i$  with (indirect) type  $x_i$  is

$$\begin{aligned} \tilde{\Pi}(x_i, b_i, \tilde{b}(\cdot)) &= \int [\tilde{v}(x_i, x_{-i}) - p^W(b_i, \tilde{b}(x_{-i}))] 1_{[b_i > \tilde{b}(x_{-i})]} dx_{-i} \\ &\quad - \int p^L(b_i, \tilde{b}(x_{-i})) 1_{[b_i < \tilde{b}(x_{-i})]} dx_{-i}. \end{aligned}$$

If  $\tilde{b}$  is discontinuous, there exists  $x^* \geq x_0$ , with

$$\limsup_{x < x^*} \tilde{b}(x) < \liminf_{x > x^*} \tilde{b}(x).$$

Consider bidders  $x_i^{+\varepsilon}$  that bids  $\beta^{+\varepsilon} = \tilde{b}(x_i^{+\varepsilon}) = \liminf_{x > x^*} \tilde{b}(x) + \varepsilon$  and  $x_i^{-\varepsilon}$  that bids  $\beta^{-\varepsilon} = \tilde{b}(x_i^{-\varepsilon}) = \limsup_{x < x^*} \tilde{b}(x) - \varepsilon$ . We will prove that, for  $\varepsilon > 0$  sufficiently small, the bid  $\beta^{-\varepsilon}$  is better than  $\beta^{+\varepsilon}$  for a bidder with type  $x_i^{+\varepsilon}$ . This will be the contradiction. The event  $[\beta^{+\varepsilon} > \tilde{b}(x_{-i})]$  is arbitrarily close to  $[\beta^{-\varepsilon} > \tilde{b}(x_{-i})]$ . So, the difference of expected utilities

$$\begin{aligned} &\int \tilde{v}(x_i^{+\varepsilon}, x_{-i}) 1_{[\beta^{+\varepsilon} > \tilde{b}(x_{-i})]} dx_{-i} - \int \tilde{v}(x_i^{+\varepsilon}, x_{-i}) 1_{[\beta^{-\varepsilon} > \tilde{b}(x_{-i})]} dx_{-i} \\ &= \int \tilde{v}(x_i^{+\varepsilon}, x_{-i}) 1_{[\beta^{+\varepsilon} > \tilde{b}(x_{-i}) \geq \beta^{-\varepsilon}]} dx_{-i} \end{aligned}$$

is arbitrarily small. On the other hand, the difference of expected payments is

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<sup>22</sup>Since our assumptions are different from theirs, we will reproduce the proof with details.

$$\begin{aligned}
& - \int \left[ p^W(\beta^{+\varepsilon}, b(x_{-i})) 1_{[\beta^{+\varepsilon} > \tilde{b}(x_{-i})]} + p^L(\beta^{+\varepsilon}, b(x_{-i})) 1_{[\beta^{+\varepsilon} < \tilde{b}(x_{-i})]} \right] dx_{-i} \\
& + \int \left[ p^W(\beta^{-\varepsilon}, b(x_{-i})) 1_{[\beta^{-\varepsilon} > \tilde{b}(x_{-i})]} + p^L(\beta^{-\varepsilon}, b(x_{-i})) 1_{[\beta^{-\varepsilon} < \tilde{b}(x_{-i})]} \right] dx_{-i} \\
& = \int \left[ (p^W(\beta^{-\varepsilon}, b(x_{-i})) - p^W(\beta^{+\varepsilon}, b(x_{-i}))) 1_{[\beta^{-\varepsilon} > \tilde{b}(x_{-i})]} \right] dx_{-i} \\
& + \int \left[ (p^L(\beta^{-\varepsilon}, b(x_{-i})) - p^L(\beta^{+\varepsilon}, b(x_{-i}))) 1_{[\beta^{-\varepsilon} < \tilde{b}(x_{-i})]} \right] dx_{-i} + r \\
& = \int \int_{\beta^{-\varepsilon}}^{\beta^{+\varepsilon}} -\partial_1 p^W(z, b(x_{-i})) 1_{[\beta^{-\varepsilon} > \tilde{b}(x_{-i})]} dz dx_{-i} \\
& + \int \int_{\beta^{-\varepsilon}}^{\beta^{+\varepsilon}} -\partial_1 p^L(z, b(x_{-i})) 1_{[\beta^{-\varepsilon} < \tilde{b}(x_{-i})]} dz dx_{-i} + r
\end{aligned}$$

where  $r$  denotes the integrals over the event  $[\beta^{+\varepsilon} > \tilde{b}(x_{-i}) \geq \beta^{-\varepsilon}]$ . Observe that the sum of the two integrals is negative and bounded away from zero, because  $\partial_1 p^W(\cdot) > 0$  or  $\partial_1 p^L(\cdot) > 0$ . So, it is not optimal for bidder  $x_i^{+\varepsilon}$  to bid  $\beta^{+\varepsilon}$  and this contradicts  $\tilde{b}$  to be an equilibrium. So,  $\tilde{b}$  is continuous. Let us prove that it is differentiable.

We have

$$\begin{aligned}
& \tilde{\Pi}(x, \tilde{b}(x), \tilde{b}(\cdot)) - \tilde{\Pi}(x, \tilde{b}(y), \tilde{b}(\cdot)) \\
& = \int_0^x \left[ \tilde{v}(x, \alpha) - p^W(\tilde{b}(x), \tilde{b}(\alpha)) \right] d\alpha - \int_x^1 p^L(\tilde{b}(x), \tilde{b}(\alpha)) d\alpha \\
& \quad - \int_0^y \left[ \tilde{v}(x, \alpha) - p^W(\tilde{b}(y), \tilde{b}(\alpha)) \right] d\alpha + \int_y^1 p^L(\tilde{b}(y), \tilde{b}(\alpha)) d\alpha \\
& = \int_x^y \left[ -\tilde{v}(x, \alpha) + p^W(\tilde{b}(y), \tilde{b}(\alpha)) - p^L(\tilde{b}(y), \tilde{b}(\alpha)) \right] d\alpha \\
& \quad + \int_0^x \left[ p^W(\tilde{b}(y), \tilde{b}(\alpha)) - p^W(\tilde{b}(x), \tilde{b}(\alpha)) \right] d\alpha \\
& \quad + \int_x^1 \left[ p^L(\tilde{b}(y), \tilde{b}(\alpha)) - p^L(\tilde{b}(x), \tilde{b}(\alpha)) \right] d\alpha.
\end{aligned}$$

Since the first integrand is continuous, by the mean value theorem, there exists  $x^*$  between  $x$  and  $y$  such that

$$\begin{aligned}
& \int_y^x \left[ \tilde{v}(x, \alpha) - p^W(\tilde{b}(y), \tilde{b}(\alpha)) + p^L(\tilde{b}(y), \tilde{b}(\alpha)) \right] d\alpha \\
& = h \left[ -\tilde{v}(x, x^*) + p^W(\tilde{b}(y), \tilde{b}(x^*)) - p^L(\tilde{b}(y), \tilde{b}(x^*)) \right]
\end{aligned}$$

Since  $p^W$  and  $p^L$  are differentiable and  $\tilde{b}$  is continuously increasing, there exist  $x^W$  and  $x^L$  between  $x$  and  $y$  such that

$$\begin{aligned} & \int_0^x \left[ p^j \left( \tilde{b}(y), \tilde{b}(\alpha) \right) - p^j \left( \tilde{b}(x), \tilde{b}(\alpha) \right) \right] d\alpha \\ &= \left[ \tilde{b}(y) - \tilde{b}(x) \right] \int_0^x \partial_1 p^j \left( \tilde{b}(x^j), \tilde{b}(\alpha) \right) d\alpha, \end{aligned}$$

where  $j = W, L$ . So, since  $\tilde{\Pi}(x, \tilde{b}(x), \tilde{b}(\cdot)) - \tilde{\Pi}(x, \tilde{b}(y), \tilde{b}(\cdot)) \geq 0$ , we have, for  $y > x$ ,

$$(31) \quad \frac{\tilde{b}(y) - \tilde{b}(x)}{y - x} \geq \frac{\left[ \tilde{v}(x, x^*) - p^W \left( \tilde{b}(y), \tilde{b}(x^*) \right) + p^L \left( \tilde{b}(y), \tilde{b}(x^*) \right) \right]}{\int_0^x \partial_1 p^W \left( \tilde{b}(x^W), \tilde{b}(\alpha) \right) d\alpha + \int_x^1 \partial_1 p^L \left( \tilde{b}(x^L), \tilde{b}(\alpha) \right) d\alpha}.$$

Analogously, interchanging  $x$  and  $y$ , and using  $\tilde{\Pi}(y, \tilde{b}(y), \tilde{b}(\cdot)) - \tilde{\Pi}(y, \tilde{b}(x), \tilde{b}(\cdot)) \geq 0$ , with  $y > x$ , we obtain, the existence of  $\hat{x}$ ,  $\hat{x}^W$  and  $\hat{x}^L$  in  $[x, y]$  such that

$$(32) \quad \frac{\tilde{b}(y) - \tilde{b}(x)}{y - x} \leq \frac{\left[ \tilde{v}(y, \hat{x}) - p^W \left( \tilde{b}(x), \tilde{b}(\hat{x}) \right) + p^L \left( \tilde{b}(x), \tilde{b}(\hat{x}) \right) \right]}{\int_0^y \partial_1 p^W \left( \hat{x}^W, \tilde{b}(\alpha) \right) d\alpha + \int_y^1 \partial_1 p^L \left( \hat{x}^L, \tilde{b}(\alpha) \right) d\alpha}$$

When  $x > y$ , we again obtain reverse inequalities. When we make  $y \rightarrow x$ , the right hand side in (31) and (32) both converge to

$$\frac{\tilde{v}(x, x) - p^W \left( \tilde{b}(x), \tilde{b}(x) \right) + p^L \left( \tilde{b}(x), \tilde{b}(x) \right)}{E_\alpha \left[ \partial_1 p^W \left( \tilde{b}(x), \tilde{b}(\alpha) \right) 1_{[\tilde{b}(x) > \tilde{b}(\alpha)]} + \partial_1 p^L \left( \tilde{b}(x), \tilde{b}(\alpha) \right) 1_{[\tilde{b}(x) < \tilde{b}(\alpha)]} \right]}.$$

So,  $\tilde{b}$  is differentiable at  $x \in (x_0, 1)$  and  $\tilde{b}'(x)$  is equal to the expression above. ■

**Proposition 3** (*Case 2*). Assume (H0)', (H1)-2 and that  $\tilde{v}$  is continuous. Let  $\tilde{b}$  be an increasing equilibrium of the indirect auction. Then,

$$(33) \quad \tilde{v}(x, x) - p^W \left( \tilde{b}(x), \tilde{b}(x) \right) + p^L \left( \tilde{b}(x), \tilde{b}(x) \right) = 0, \forall x \in (x_0, 1).$$

**Proof.** The proof is based on the proof of Theorem 3 of Maskin and Riley (1984). Given  $b, b'$ , let us define the function  $h$  as

$$h(z) \equiv p^W(b, z) - p^L(b', z).$$

Since  $\partial_1 p^W = \partial_1 p^L \equiv 0$  and  $\partial_2(p^W - p^L) > 0$ ,  $h$  does not depend on  $b$  or  $b'$  and is differentiable and increasing.

By contradiction, assume that (33) is false, that is, there exists  $x^* \in (x_0, 1)$  such that

$$(34) \quad \tilde{v}(x^*, x^*) > h \left( \tilde{b}(x^*) \right).$$

Because  $\tilde{v}$  is continuous and  $\tilde{b}$  is increasing, for sufficiently small  $\delta > 0$ , we have

$$\tilde{v}(x^* - \delta, x^* - \delta) > h \left( \tilde{b}(x^*) \right) > h \left( \tilde{b}(x^* - \delta) \right).$$

Since the set of discontinuity points of  $\tilde{b}$  is enumerable, we may assume that (34) holds for a point  $x^*$  where  $\tilde{b}$  is continuous. Thus, for sufficiently small  $\varepsilon$  and  $\delta > 0$ ,  $\forall \alpha \in [x^*, x^* + \varepsilon]$ ,

$$\tilde{v}(x^*, \alpha) > h(\tilde{b}(x^*) + \delta) > h(\tilde{b}(\alpha)).$$

Consider the following difference:

$$\begin{aligned} & \tilde{\Pi}(x^*, \tilde{b}(x^* + \varepsilon), \tilde{b}(\cdot)) - \tilde{\Pi}(x^*, \tilde{b}(x^*), \tilde{b}(\cdot)) \\ &= \int_0^{x^* + \varepsilon} \left[ \tilde{v}(x^*, \alpha) - p^W(\tilde{b}(x^* + \varepsilon), \tilde{b}(\alpha)) \right] d\alpha - \int_{x^* + \varepsilon}^1 p^L(\tilde{b}(x^* + \varepsilon), \tilde{b}(\alpha)) d\alpha \\ & \quad - \int_0^{x^*} \left[ \tilde{v}(x^*, \alpha) - p^W(\tilde{b}(x^*), \tilde{b}(\alpha)) \right] d\alpha + \int_{x^*}^1 p^L(\tilde{b}(x^*), \tilde{b}(\alpha)) d\alpha \\ &= \int_{x^*}^{x^* + \varepsilon} \left[ \tilde{v}(x^*, \alpha) - h(\tilde{b}(\alpha)) \right] d\alpha > 0 \end{aligned}$$

where we used the property  $\partial_1 p^W = \partial_1 p^L \equiv 0$  in order to obtain the last equality. This contradicts the optimality of  $\tilde{b}(x^*)$  for  $x^*$ .

Analogously, if we assume that there is a  $x^* \in (x_0, 1)$  such that  $\tilde{v}(x^*, x^*) < h(\tilde{b}(x^*))$  and that  $x^*$  is a point of continuity of  $\tilde{b}$ , we have for  $\varepsilon > 0$  sufficiently small,

$$\begin{aligned} & \tilde{\Pi}(x^*, \tilde{b}(x^*), \tilde{b}(\cdot)) - \tilde{\Pi}(x^*, \tilde{b}(x^* - \varepsilon), \tilde{b}(\cdot)) \\ &= \int_{x^* - \varepsilon}^{x^*} \left[ \tilde{v}(x^*, \alpha) - h(\tilde{b}(\alpha)) \right] d\alpha < 0. \end{aligned}$$

This completes the proof of (33). So, we have

$$(35) \quad \tilde{b}(x) = h^{-1}(\tilde{v}(x, x)),$$

which shows that  $\tilde{b}$  is continuous. Moreover,  $\tilde{b}$  is increasing if and only if  $x \mapsto \tilde{v}(x, x)$  is also increasing. ■

Now, we will analyze the equilibrium existence in both cases 1 and 2. Instead of assuming that  $\tilde{v}$  is continuous, as we did in the last two propositions, we will assume directly its consequence, that is, we suppose that there exists a function  $\tilde{b}$  that satisfies the following:

*Case 1:*  $\partial_1 p^W(\cdot) > 0$  or  $\partial_1 p^L(\cdot) > 0$ ,  $\tilde{b}$  is differentiable and

$$\tilde{b}'(x) = \frac{\tilde{v}(x, x) - p^W(\tilde{b}(x), \tilde{b}(x)) + p^L(\tilde{b}(x), \tilde{b}(x))}{E_\alpha \left[ \partial_1 p^W(\tilde{b}(x), \tilde{b}(\alpha)) 1_{[\tilde{b}(x) > \tilde{b}(\alpha)]} + \partial_1 p^L(\tilde{b}(x), \tilde{b}(\alpha)) 1_{[\tilde{b}(x) < \tilde{b}(\alpha)]} \right]}.$$

*Case 2:*  $\partial_1 p^W = \partial_1 p^L = 0$ ,  $\partial_2(p^W - p^L) > 0$ ,  $\tilde{b}$  is continuous, and

$$x \in (x_0, 1) \Rightarrow \tilde{v}(x, x) = h(\tilde{b}(x)) = p^W(\tilde{b}(x), \tilde{b}(x)) - p^L(\tilde{b}(x), \tilde{b}(x)).$$

Observe that we do not assume that  $\tilde{b}$  is increasing. This is so because this is exactly the setting of Theorem 4. To treat non-increasing  $\tilde{b}$ , we define the following:

**Modified Auction** - The bidder submits a type  $y \in [0, 1]$ . In any event, the payment is determined as if the bidder has submitted the bid  $\tilde{b}(y)$ . The bidder wins against opponents who announce types below  $y$  and loses against opponents who announce types above  $y$ . If there is a tie, the object is given with probability  $1/2$  for each bidder.

Observe that if  $\tilde{b}$  is increasing, the modified auction is simply the indirect auction. If  $\tilde{b}$  is not increasing, the difference is that the winning events are not determined by  $\tilde{b}$  but by the announced type  $y$ . The rule of the modified auction implies the following interim payoff:

$$\hat{\Pi}(x, y) = \begin{cases} \int_0^y [\tilde{v}(x, \alpha) - p^W(\tilde{b}(y), \tilde{b}(\alpha))] d\alpha - \int_y^1 p^L(\tilde{b}(y), \tilde{b}(\alpha)) d\alpha, & \text{if } y \geq x_0 \\ 0 & \text{if } y < x_0 \end{cases}$$

We can simplify the above expression to

$$(36) \quad \hat{\Pi}(x, y) = \begin{cases} \int_0^y \tilde{v}(x, \alpha) d\alpha - \hat{p}(y), & \text{if } y \geq x_0 \\ 0 & \text{if } y < x_0 \end{cases}$$

where

$$\hat{p}(y) \equiv \begin{cases} \int_0^y p^W(\tilde{b}(y), \tilde{b}(\alpha)) d\alpha + \int_y^1 p^L(\tilde{b}(y), \tilde{b}(\alpha)) d\alpha, & \text{if } y \geq x_0 \\ 0, & \text{if } y < x_0 \end{cases}$$

In case 1,  $\tilde{b}$ ,  $p^W$  and  $p^L$  are differentiable on  $(x_0, 1)$ ,  $\hat{p}$  and  $\hat{\Pi}$  are also differentiable. So, for every  $y \in (x_0, 1)$ , we have

$$\hat{p}'(y) = \partial_y \left\{ \int_0^y \tilde{v}(x, \alpha) d\alpha - \hat{\Pi}(x, y) \right\} = \tilde{v}(x, y) - \partial_y \hat{\Pi}(x, y).$$

Truth-telling is always optimal if

$$(37) \quad \hat{\Pi}(x, x) - \hat{\Pi}(x, y) \geq 0.$$

In case 1, this is equivalent to

$$\int_y^x \partial_y \hat{\Pi}(x, \alpha) d\alpha \geq 0$$

if  $x, y \geq x_0$ . Also, if  $x, y \geq x_0$ ,  $\partial_y \hat{\Pi}(x, y) |_{y=x}$  must be zero, so that

$$(38) \quad \partial_y \hat{\Pi}(x, y) |_{y=x} = 0 \Rightarrow \hat{p}'(x) = \tilde{v}(x, x).$$

Indeed, these are simply the second- and the first-order conditions, respectively. So, for  $y \geq x_0$ ,

$$\hat{p}(y) = \int_{x_0}^y \tilde{v}(\alpha, \alpha) d\alpha + \hat{p}(x_0).$$



Now, let us turn to case 2. Since  $\tilde{b}$  is only continuous,  $\hat{p}$  is not necessarily differentiable. Nevertheless, if  $y \geq x_0$ ,

$$\begin{aligned}
\hat{p}(y) &= \int_0^y p^W(\tilde{b}(y), \tilde{b}(\alpha)) d\alpha + \int_y^1 p^L(\tilde{b}(y), \tilde{b}(\alpha)) d\alpha \\
&= \int_{x_0}^y \left[ p^W(\tilde{b}(y), \tilde{b}(\alpha)) - p^L(\tilde{b}(y), \tilde{b}(\alpha)) \right] d\alpha + \hat{p}(x_0) \\
&= \int_{x_0}^y h(\tilde{b}(\alpha)) d\alpha + \hat{p}(x_0) \\
&= \int_{x_0}^y \tilde{v}(\alpha, \alpha) d\alpha + \hat{p}(x_0).
\end{aligned}$$

Observe that the expression above is exactly the same of case 1. For  $y < x_0$ , the payment is zero. For  $y = x_0$ ,  $\hat{p}(y)$  is obtained from (29):

$$\int_0^{x_0} \tilde{v}(x_0, \alpha) d\alpha - \int_0^{x_0} p^W(\tilde{b}(x_0), \tilde{b}(\alpha)) d\alpha + \int_{x_0}^1 p^L(\tilde{b}(x_0), \tilde{b}(\alpha)) d\alpha = 0.$$

So, we have proved the following:

**Proposition 4** (Payment Rule). Assume (H0)', (H1) and (H2)'. Then, both for case 1 or case 2, we have

$$(39) \quad \hat{p}(y) = \begin{cases} \hat{p}(x_0) + \int_{x_0}^y \tilde{v}(\alpha, \alpha) d\alpha, & \text{if } y > x_0 \\ \int_0^{x_0} \tilde{v}(x_0, \alpha) d\alpha, & \text{if } y = x_0 \\ 0, & \text{if } y < x_0 \end{cases}$$

Now, we turn to the equilibrium existence.

**Proposition 5** (Equilibrium). Assume (H0)', (H1), (H2)' and (39). Then, truth-telling is equilibrium of the modified auction if and only if, for all  $x, y \in [0, 1]$ ,

$$(40) \quad \begin{cases} \int_y^x [\tilde{v}(x, \alpha) - \tilde{v}(\alpha, \alpha)] d\alpha \geq 0, & \text{if } x, y \geq x_0 \\ \int_{x_0}^x [\tilde{v}(x, \alpha) - \tilde{v}(\alpha, \alpha)] d\alpha + \int_0^{x_0} [\tilde{v}(x, \alpha) - \tilde{v}(x_0, \alpha)] d\alpha \geq 0, & \text{if } x \geq x_0 > y \\ 0 \geq \int_{x_0}^y [\tilde{v}(x, \alpha) - \tilde{v}(\alpha, \alpha)] d\alpha + \int_0^{x_0} [\tilde{v}(x, \alpha) - \tilde{v}(x_0, \alpha)] d\alpha & \text{if } y \geq x_0 > x \end{cases}$$

**Proof.** Given (39), the optimality condition for truth-telling, namely  $\hat{\Pi}(x, x) - \hat{\Pi}(x, y) \geq 0$ , is equivalent to

$$\begin{aligned}
& \int_0^x \tilde{v}(x, \alpha) d\alpha - \int_{x_0}^x \tilde{v}(\alpha, \alpha) d\alpha - \hat{p}(x_0) \\
& - \int_0^y \tilde{v}(x, \alpha) d\alpha + \int_{x_0}^y \tilde{v}(\alpha, \alpha) d\alpha + \hat{p}(x_0) \\
(41) \quad & = \int_y^x [\tilde{v}(x, \alpha) - \tilde{v}(\alpha, \alpha)] d\alpha \geq 0
\end{aligned}$$

if  $x, y \geq x_0$ . The other cases are immediate. ■

As we have said before, if  $\tilde{b}$  is increasing, the modified auction is just the original (unmodified) auction. Then, we have:

**Corollary 1.** Assume (H0)', (H1), (H2)' and that  $\tilde{b}$  is increasing and implies (39). Then, if (40) holds,  $\tilde{b}$  is equilibrium of the indirect auction.

Observe that Corollary 1 does not require  $\tilde{v}$  to be continuous.

## Appendix C - Proofs of the Theorems

### Proof of Theorem 1.

(i) If  $b \in \mathcal{S}$ , it defines a conjugation  $P^b$  by (2). The bid  $b(t_i) = \beta$  is optimal for bidder  $t_i$  against the strategy  $b(\cdot)$  of the opponents. This and the fact that  $\partial_b \Pi(s, b(s)) = 0$  imply that

$$\begin{aligned} E[v(t_i, \cdot) | t_i = s, b_{(-i)}(t_{-i}) = \beta] \\ = p^W(\beta, \beta) - p^L(\beta, \beta) - \frac{Et_{-i} [\partial_{b_i} p^W 1_{[b_i > \mathbf{b}_{(-i)}]} + \partial_{b_i} p^L 1_{[b_i < \mathbf{b}_{(-i)}]}]}{f_{b_{(-i)}}(\beta)}. \end{aligned}$$

Observe that the right-hand side does not depend on  $s$  (it depends on it only by the fact that  $\beta = b(s)$  is the optimum bid for such bidder). Thus, the left-hand side has to be the same for all  $s$  that are bidding the same bid in equilibrium, which implies that (10) holds.

(ii) If  $b(t_i)$  maximizes  $\Pi(t_i, c)$  for  $t_i$ , and  $P(t'_i) = P(t_i)$ , then  $b(t'_i) = b(t_i)$ . Then,  $b(t_i)$  maximizes  $\tilde{\Pi}(P(t'_i), c)$  for all  $t'_i$  such that  $P(t'_i) = P(t_i)$ , from the definition of  $\tilde{\Pi}(P(t_i), c)$  given by (6). In other words,  $\tilde{b}(x) = \left(\tilde{P}^b\right)^{-1}(x) = b(P^{-1}(x))$  is the equilibrium of the indirect auction.

If  $\tilde{v}$  is continuous, we appeal to the results of Appendix B. Propositions 2 and 3 prove (iii), Proposition 4 proves (iv) and Proposition 5 gives (v). ■

**Proof of Theorem 2.** Corollary 1 of Appendix B proves that conditions (ii) and (iii) are sufficient for  $\tilde{b}$  to be the equilibrium of the indirect auction. Now, Proposition 1 proves that condition (i)' implies that for all  $s$  such that  $P(s) = x$ ,  $\tilde{\Pi}(x, c) = \Pi(s, c)$  (see (9)). Now, if we put  $b(s) = \tilde{b}(P(s))$ , then

$$\begin{aligned} \Pi(s, b(s)) &= \tilde{\Pi}(P(s), \tilde{b}(P(s))) \text{ and} \\ \Pi(s, c) &= \tilde{\Pi}(P(s), c). \end{aligned}$$

But this is sufficient to show the equilibrium existence in the direct auction, since  $\tilde{b}$  is the equilibrium in the indirect auction, which implies that

$$\tilde{\Pi}(P(s), \tilde{b}(P(s))) \geq \tilde{\Pi}(P(s), c),$$

for all  $c \in \mathbb{R}$ . If  $\tilde{v}$  is continuous,  $\tilde{\Pi}(x, c)$  is differentiable at all  $c \in \mathbb{R}$ . This concludes the proof. ■

Through the proof of Theorem 3, we will make successive use of the following fact:

**Lemma 2.** Assume (H1), (H2) and (H3). For any  $\sigma$ -field  $\Sigma$  on  $S^{N-1}$ , we have

$$\begin{aligned} & \exists t_{-i} : v(s', t_{-i}) > v(s, t_{-i}) \\ & \Leftrightarrow \forall t_{-i} : v(s', t_{-i}) > v(s, t_{-i}) \\ & \Leftrightarrow E[v(t_i, t_{-i}) | t_i = s', \Sigma] > E[v(t_i, t_{-i}) | t_i = s, \Sigma], \text{ a.s.} \end{aligned}$$

**Proof.** (H3) gives the first equivalence. By (H2),  $v$  is continuous over a compact. So, if  $\forall t_{-i} : v(s', t_{-i}) > v(s, t_{-i})$ , there is  $\delta > 0$  so that  $d(t_{-i}) \equiv v(s', t_{-i}) - v(s, t_{-i}) - \delta \geq 0$  for all  $t_{-i}$ . Then, for any  $\Sigma$ ,  $E[d(t_{-i}) | \Sigma] \geq 0$  almost surely.<sup>23</sup> This implies that  $E[v(t_i, t_{-i}) | t_i = s', \Sigma] > E[v(t_i, t_{-i}) | t_i = s, \Sigma]$ , a.s. On the other hand,  $E[v(t_i, t_{-i}) | t_i = s', \Sigma] > E[v(t_i, t_{-i}) | t_i = s, \Sigma]$  a.s. implies that  $\exists t_{-i} : v(s', t_{-i}) > v(s, t_{-i})$ . ■

**Proof of Theorem 3.** *Equilibrium Existence.* If we define  $P$  by (19), it is a conjugation. Let us prove that it satisfies condition (i)' of Theorem 2. If for some  $x, y$  and  $s$ , such that  $P(s) = x$ , we have

$$\tilde{v}(x, y) = E[v(t_i, t_{-i}) | P(t_i) = x, P_{(-i)}(t_{-i}) = y] < E[v(t_i, t_{-i}) | t_i = s, P_{(-i)}(t_{-i}) = y],$$

then, for at least one  $t_{-i}$  and  $s'$ ,  $P(s') = x$ ,  $v(s, t_{-i}) > v(s', t_{-i})$ . But then, by (H3),  $v(s, t_{-i}) > v(s', t_{-i})$  for all  $t_{-i}$  which implies  $v^1(s) > v^1(s')$  and  $P(s) > P(s')$ , a contradiction with the assumption that  $P(s) = P(s') = x$ . So, condition (i)' is satisfied.

Let us prove condition (ii) of Theorem 2. If  $x > y$ , for all  $t_i$  and  $t'_i$  such that  $P(t'_i) = x$  and  $P(t_i) = y$ , we have  $v(t'_i, t_{-i}) > v(t_i, t_{-i})$  for all  $t_{-i}$ , by (H3). Then, for all  $z \in [0, 1]$ ,

$$\begin{aligned} \tilde{v}(x, z) & \equiv E[v(t_i, t_{-i}) | P(t_i) = x, P_{(-i)}(t_{-i}) = z] \\ & > E[v(t_i, t_{-i}) | P(t_i) = y, P_{(-i)}(t_{-i}) = z] = \tilde{v}(y, z). \end{aligned}$$

Then, if  $y < \alpha < x$ ,  $\tilde{v}(x, \alpha) - \tilde{v}(\alpha, \alpha) > 0$  and we have:

$$\int_y^x [\tilde{v}(x, \alpha) - \tilde{v}(\alpha, \alpha)] d\alpha \geq 0.$$

Now if  $x < \alpha < y$ , we have  $\tilde{v}(x, \alpha) - \tilde{v}(\alpha, \alpha) < 0$  so that condition (ii) is satisfied. Since our assumption is the condition (iii) of Theorem 2, this implies the existence of equilibrium, with the equilibrium bidding function given by  $b = \tilde{b} \circ P$ .

*Sufficiency.* Conditions (i)' and (ii) of the Theorem 2 was shown in the first part, above. Proposition 4 in appendix B proves condition (iii) of Theorem 2. Then, there exists a equilibrium  $b = \tilde{b} \circ P$ . Since  $\tilde{v}$  is continuous, Theorem 2 shows the existence of  $\partial_b \Pi(s, b(s))$  for all  $s$ .

*Necessity.* According to Theorem 1, given a  $b \in \mathcal{S}$ , the associated conjugation  $P^b$  (given by (2)) is such that for all  $s \in (P^b)^{-1}(x)$ ,

$$E[v(t_i, t_{-i}) | P^b(t_i) = x, P_{(-i)}^b(t_{-i}) = x] = E[v(t_i, t_{-i}) | t_i = s, P_{(-i)}^b(t_{-i}) = x].$$

<sup>23</sup>See, for instance, Kallenberg (2002), Theorem 6.1, p. 104.

If  $P^b(s) = P^b(s')$  and there is some  $t_{-i}$  such that  $v(s, t_{-i}) < v(s', t_{-i})$ , Lemma 2 implies that

$$E[v(t_i, t_{-i}) | t_i = s, P_{(-i)}^b(t_{-i}) = x] < E[v(t_i, t_{-i}) | t_i = s', P_{(-i)}^b(t_{-i}) = x],$$

which contradicts the previous equality between the conditional expectations. We conclude that

$$(42) \quad P^b(s) = P^b(s') \Rightarrow v(s, t_{-i}) = v(s', t_{-i}) \text{ for all } t_{-i}.$$

Let us define  $\tilde{v}^1(x)$  as  $E[v(t_i, t_{-i}) | P^b(t_i) = x]$  and prove that it is non-decreasing. Suppose by absurd that there exist  $x$  and  $y$ ,  $x > y$ , such that  $\tilde{v}^1(x) < \tilde{v}^1(y)$ .

First, we claim that for all  $t_i$  and  $t'_i$  such that  $P^b(t_i) = x$  and  $P^b(t'_i) = y$ , we have  $v(t_i, t_{-i}) < v(t'_i, t_{-i})$  for all  $t_{-i}$ . Otherwise,  $v(t_i, t_{-i}) \geq v(t'_i, t_{-i})$  for some  $t_{-i}$  and, by (H3),  $v(t_i, t'_{-i}) \geq v(t'_i, t'_{-i})$  for all  $t'_{-i}$ . Then, Lemma 2 and (42) would imply that  $\tilde{v}^1(x) = E[v(t_i, t_{-i}) | P^b(t_i) = x] \geq E[v(t_i, t_{-i}) | P^b(t_i) = y] = \tilde{v}^1(y)$ , a contradiction with our (absurd) assumption. Thus, the claim is proved.

This claim and Lemma 2 imply that

$$\begin{aligned} \tilde{v}(x, z) &\equiv E[v(t_i, t_{-i}) | P^b(t_i) = x, P_{(-i)}^b(t_{-i}) = z] \\ &< E[v(t_i, t_{-i}) | P^b(t_i) = y, P_{(-i)}^b(t_{-i}) = z] = \tilde{v}(y, z), \end{aligned}$$

for all  $z \in [0, 1]$ , a.s. Thus,

$$\int_y^x [\tilde{v}(x, \alpha) - \tilde{v}(y, \alpha)] d\alpha < 0.$$

By condition (v) of Theorem 1, we also have that

$$\int_y^x [\tilde{v}(y, \alpha) - \tilde{v}(\alpha, \alpha)] d\alpha \leq 0.$$

Summing up these two integrals, we obtain

$$\int_y^x [\tilde{v}(x, \alpha) - \tilde{v}(\alpha, \alpha)] d\alpha < 0,$$

which contradicts condition (v) of Theorem 1. This contradiction establishes that  $x > y \Rightarrow \tilde{v}^1(x) \geq \tilde{v}^1(y)$ .

Suppose now that there exists  $x > y$  such that  $\tilde{v}^1(x) = \tilde{v}^1(y)$ . Then, the monotonicity of  $\tilde{v}^1$  (just proved) gives

$$(43) \quad \forall \phi \in [y, x], \tilde{v}^1(\phi) = \tilde{v}^1(x) = \tilde{v}^1(y).$$

Let  $S' = \{s \in S : \tilde{b}(y) \leq b(s) < \tilde{b}(x)\}$ . From (2), for all  $s \in S'$ ,  $P^b(s) \in [y, x]$ . Then, (42) and (43) imply that  $s \in S' \Rightarrow v^1(s) = \tilde{v}^1(x)$ . Assumption (H3) requires that  $\mu(S') = 0$ . Observe that  $S' = A \setminus B$ , where  $A \equiv \{s \in S : b(s) < \tilde{b}(x)\}$  and  $B = \{s \in S : b(s) < \tilde{b}(y)\}$ . But then,  $\mu(A) = \mu(B)$ . However, from the definition of  $\tilde{b}$  as the inverse of  $\tilde{P}^b$ , we have the following:

$$0 < x - y = \tilde{P}^b(\tilde{b}(x)) - \tilde{P}^b(\tilde{b}(y)) = (\mu(A))^{N-1} - (\mu(B))^{N-1},$$

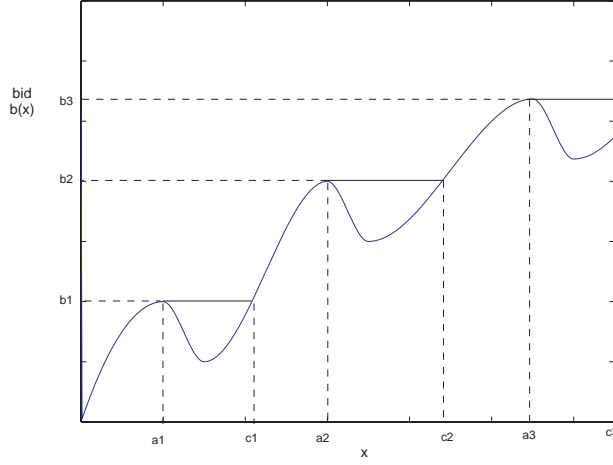


FIGURE 5. Indirect Equilibrium Bidding Function

which is a contradiction. So, we have proved that  $x = P^b(s') > P^b(s) = y$  implies  $v^1(s') = \tilde{v}^1(x) > \tilde{v}^1(y) = v^1(s)$  and  $P^b(s') = P^b(s)$  implies  $v^1(s') = v^1(s)$ . In other words,  $P^b(s') \begin{smallmatrix} \leq \\ \geq \end{smallmatrix} P^b(s)$  if and only if  $v^1(s') \begin{smallmatrix} \leq \\ \geq \end{smallmatrix} v^1(s)$  which allows us to conclude that

$$P^b(t_i) = \Pr \{t_{-i} \in T_{-i} = S^{N-1} : v^1(t_j) < v^1(t_i), j \neq i\},$$

as we have defined in (19). In other words, the conjugation is unique.

Now,  $\tilde{v}$  and  $\tilde{b}$  in Theorem 1 are exactly those defined in the statement of Theorem 3. So, Theorem 1 implies the claims about  $\tilde{b}$ .

*Uniqueness.* Since  $\tilde{v}$  is continuous, Propositions 2 and 3 in Appendix B says that any equilibrium  $\tilde{b}$  satisfy the conditions given. If there is just one  $\tilde{b}$  that satisfy such conditions, then the equilibrium of the indirect auction is unique. Since the previous step (necessity) shows that the conjugation is unique, the equilibrium of the direct auction is unique. ■

**Proof of Theorem 4.** If  $\tilde{b}$  is strictly increasing, then  $b = \tilde{b} \circ P$  is an equilibrium of direct auction, by Theorem 3.

So, we have to show that an equilibrium exists if  $\tilde{b}$  is not increasing. For future use, remember that in the first part of the proof of Theorem 3, we have established conditions (i)' and (ii) of Theorem 2 and that

$$(44) \quad x > y \Rightarrow \tilde{v}(x, z) > \tilde{v}(y, z), \forall z \in [0, 1].$$

Let us define  $\bar{b}(x) = \sup_{\alpha \in [0, x]} \tilde{b}(\alpha)$ . As we discussed after the statement of Theorem 4, this is just one of the possible specification for the equilibrium bidding function. The only exception is when the tie is to occur including the highest bidder. In such a case, it is mandatory to have the bid of tying bidders following the above definition. The reason will become clear in the sequel.

Remember that  $\tilde{b}$  is absolutely continuous. Then, there is an enumerable set of intervals  $[a_k, c_k]$  where  $\bar{b}(x)$  is constant. Let  $b_k \equiv \bar{b}(x)$  for  $x \in [a_k, c_k]$ . (See Figure 5.)

Therefore, there is a tie among the indirect types in  $[a_k, c_k]$  for the bidding function  $\bar{b}$ . Let  $b_k$  be the specified bid for indirect types in  $[a_k, c_k]$ , that is,  $\bar{b}([a_k, c_k]) = \{b_k\}$ . The tie is solved by an all-pay auction among tying bidders.

The unique information that bidders have for the second auction is that there is a tie in  $b_k$ , that is,  $P_{(-i)}(t_{-i}) \in [a_k, c_k]$ .

By the definition of  $P$  in (19),  $P_{(-i)}$  satisfies the following:

$$\Pr(\{t_{-i} \in S^{N-1} : P_{(-i)}(t_{-i}) < x\} | P_{(-i)}(t_{-i}) \in [a_k, c_k]) = \frac{x - a_k}{c_k - a_k}.$$

So, in the tie-breaking auction, the (direct) type  $t_i$  of bidder  $i$  is competing against players  $t_j$  in the set  $\{s \in S : P(s) \in [a_k, c_k]\}$  and the equilibrium is to bid the increasing function<sup>24</sup>

$$\tilde{b}^2(x) = \frac{1}{c_k - a_k} \int_{a_k}^x \tilde{v}(\alpha, \alpha) d\alpha.$$

Indeed, from condition (ii) of Theorem 2, we have that

$$\begin{aligned} & \frac{1}{c_k - a_k} \left[ \int_{a_k}^x \tilde{v}(x, \alpha) d\alpha - \int_{a_k}^x \tilde{v}(\alpha, \alpha) d\alpha \right] \\ & \geq \frac{1}{c_k - a_k} \left[ \int_{a_k}^y \tilde{v}(x, \alpha) d\alpha - \int_{a_k}^y \tilde{v}(\alpha, \alpha) d\alpha \right] \end{aligned}$$

for any  $x, y \in [a_k, c_k]$ .

Thus, in the whole auction, the bidder of indirect type  $x \in [a_k, c_k]$  who follows the strategy  $\bar{b}(x)$  and, in case of a tie, the above strategy, will receive the expected payoff

$$\begin{aligned} & \int_0^{a_k} [\tilde{v}(x, \alpha) - \tilde{v}(\alpha, \alpha)] d\alpha + (c_k - a_k) \left\{ \frac{1}{c_k - a_k} \int_{a_k}^x [\tilde{v}(x, \alpha) - \tilde{v}(\alpha, \alpha)] d\alpha \right\} \\ & = \int_0^x [\tilde{v}(x, \alpha) - \tilde{v}(\alpha, \alpha)] d\alpha \end{aligned}$$

Deviation in the second auction is suboptimal. By deviating from  $\bar{b}$ , but bidding in the range of  $\bar{b}$ , that is, bidding  $\bar{b}(y) \neq \bar{b}(x)$ , he will get

$$\tilde{\Pi}_i(x, \bar{b}(y)) = \int_0^y [\tilde{v}(x, \alpha) - \tilde{v}(\alpha, \alpha)] d\alpha,$$

if  $\bar{b}(y)$  is not a bid with positive probability. This cannot be profitable by condition (ii). If it is a bid with positive probability, the second stage will be again an all-pay auction, where the bidder cannot improve its payoff, again by condition (ii).

Now, if  $x$  bids  $\beta < \inf_{x \in [0, 1]} \bar{b}(x)$ , then his payoff will be

$$\int_0^1 p^L(\beta, \bar{b}(\alpha)) d\alpha \leq 0,$$

because  $p^L \leq 0$ . Therefore, this deviation cannot be profitable.

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<sup>24</sup>It is increasing because  $\tilde{v}$  is positive.

If  $x$  bids  $\beta > \sup_{x \in [0,1]} \bar{b}(x) = \tilde{b}(\bar{x}) \geq \tilde{b}(1)$ , for some  $\bar{x}$ . Since  $\partial_1 p^W(\cdot) \geq 0$ ,  $p^W(\beta, \bar{b}(z)) \geq p^W(\tilde{b}(1), \bar{b}(z))$ . Then,

$$\int_0^1 p^W(\beta, \bar{b}(\alpha)) d\alpha \geq \int_0^1 p^W(\tilde{b}(1), \bar{b}(\alpha)) d\alpha = \int_0^1 \tilde{v}(\alpha, \alpha) d\alpha.$$

Then, the payoff of the bidder with indirect type  $x$  that bids  $\beta$  will be

$$\begin{aligned} & \int_0^1 [\tilde{v}(x, \alpha) - p^W(\beta, \bar{b}(\alpha))] d\alpha \\ & \leq \int_0^1 [\tilde{v}(x, \alpha) - \tilde{v}(\alpha, \alpha)] d\alpha \\ & = \int_0^x [\tilde{v}(x, \alpha) - \tilde{v}(\alpha, \alpha)] d\alpha + \int_x^1 [\tilde{v}(x, \alpha) - \tilde{v}(\alpha, \alpha)] d\alpha \\ & < \int_0^x [\tilde{v}(x, \alpha) - \tilde{v}(\alpha, \alpha)] d\alpha, \end{aligned}$$

where the last inequality comes from (44). Thus, the deviation to  $\beta$  is unprofitable.

In Theorem 3, we also proved condition (i)'. Then, the equilibrium in the indirect auction gives the equilibrium for the direct one. ■

**Proof of Theorem 5.** If  $y$  is an indirect type that is not involved in ties, the payment is given by

$$\int_0^y p^W(\tilde{b}(y), \tilde{b}(\alpha)) d\alpha + \int_y^1 p^L(\tilde{b}(y), \tilde{b}(\alpha)) d\alpha = \int_0^y \tilde{v}(\alpha, \alpha) d\alpha.$$

If  $x \in [a_k, c_k]$ , in the notation of the previous proof, the expected payment of  $x$  will be

$$\int_0^{a_k} \tilde{v}(\alpha, \alpha) d\alpha + (c_k - a_k) \left\{ \frac{1}{c_k - a_k} \int_{a_k}^x \tilde{v}(\alpha, \alpha) d\alpha \right\} = \int_0^x \tilde{v}(\alpha, \alpha) d\alpha.$$

So, if the equilibrium specified in the proof of Theorem 4 is followed, the expected payment does not depend on the auction format. ■

## Appendix D - Proofs for the Examples

### Generalization of Example 1.

As in Example 1, consider a symmetric auction with two bidders whose utility functions are given by:

$$v(t_i, t_{-i}) = t_i + \alpha(t_i)t_{-i}$$

where  $\alpha : [0, 1] \rightarrow \mathbb{R}$  and  $t_i$  is uniformly distributed on  $[0, 1]$ . If there are only two pooling types, that is, types which bid the same for each equilibrium bidding, then, for each  $i$  and  $t$ , the respective pooling type of  $t$ ,  $\varphi = \varphi(t)$ , in a symmetric equilibrium  $(b^*, b^*)$  satisfies the condition (i) of Theorem 1:

$$t + \alpha(t)E[t_2 | b^*(t) = b^*(t_2)] = \varphi + \alpha(\varphi)E[t_2 | b^*(t) = b^*(t_2)].$$

Since  $E[t_2|b^*(t) = b^*(t_2)] = (t + \varphi)/2$ , because of the symmetry and the uniform distribution, then  $\varphi$  is the implicit solution of

$$(t + \varphi)(\alpha(\varphi) - \alpha(t)) = 2(t - \varphi).$$

**Claim.** Assume the following conditions:

- (i)  $\alpha$  is differentiable, decreasing and convex such that  $\alpha(0) - \alpha(\bar{v}) = 2$ ;
- (ii)  $\alpha'$  is strictly convex and  $\alpha'(x) \geq -1/x$  for all  $x \in (0, 1]$ ;

Then there exists an U-shaped symmetric equilibrium.

**Proof.** Define the conjugation

$$P(t) = \frac{\varphi(t) - t}{2}.$$

It is easy to see that  $P$  is decreasing.

Define

$$\begin{aligned} \tilde{v}(x, y) &\equiv E[v(t_1, t_2) | P(t_1) = x, P(t_2) = y] \\ &= \frac{x + \varphi(x)}{2} + \frac{\alpha(x) + \alpha(\varphi(x))}{2} \frac{y + \varphi(y)}{2} \end{aligned}$$

Observe that  $v(t_1, t_2)$  is of the form  $\sum_{k=1}^2 f_k(t_i) g_k(t_{-i})$ . Therefore, by Remark 2 and Theorem 2, the bidding function is an equilibrium if

$$x \geq y \Rightarrow \tilde{v}(x, y) \geq \tilde{v}(y, y).$$

Dividing by  $(y + \varphi(y))/2$ , we rewrite the above condition as

$$(45) \quad \frac{x + \varphi(x)}{y + \varphi(y)} + \frac{\alpha(x) + \alpha(\varphi(x))}{2} \geq 1 + \frac{\alpha(y) + \alpha(\varphi(y))}{2}$$

For each  $w \in (0, 1]$ , define  $g_w(z) = \frac{z}{w} + \alpha(z)$ , for  $z \in [w, 1]$ . It is easy to see that  $g$  is non-decreasing (because  $g'_w(z) = \frac{1}{w} + \alpha'(z) \geq \frac{1}{w} - \frac{1}{z} \geq 0$ ). Let  $y$  and  $w = \frac{y + \varphi(y)}{2}$  be fixed and take  $x \geq y$ . Since  $\alpha$  is convex,

$$\begin{aligned} \frac{x + \varphi(x)}{y + \varphi(y)} + \frac{\alpha(x) + \alpha(\varphi(x))}{2} &\geq \frac{x + \varphi(x)}{y + \varphi(y)} + \alpha\left(\frac{x + \varphi(x)}{2}\right) \\ &= g_w\left(\frac{x + \varphi(x)}{2}\right), \end{aligned}$$

if  $x + \varphi(x) \geq y + \varphi(y)$ . So, to show (45), it is enough to show that  $t + \varphi(t)$  is non-increasing in  $t$  or, equivalently,  $\varphi'(t) \leq -1$ .

The implicit derivative of  $\varphi$  with respect to  $t$  is:

$$\varphi'(t) = \frac{\alpha(t) - \alpha(\varphi(t)) + (t + \varphi(t))\alpha'(t) + 2}{\alpha(\varphi(t)) - \alpha(t) + (t + \varphi(t))\alpha'(\varphi) + 2}.$$

If  $\varphi'(t) \leq -1$ , then the numerator and denominator of the fraction above should have opposite sign. Without loss of generality (because  $\varphi \circ \varphi(t) = t$ ), we can assume that the denominator is negative and  $\varphi > t$ . Thus,

$$\varphi'(t) \leq -1 \Leftrightarrow \frac{\alpha'(t) + \alpha'(\varphi)}{2} \geq \frac{\alpha(\varphi) - \alpha(t)}{\varphi - t}.$$



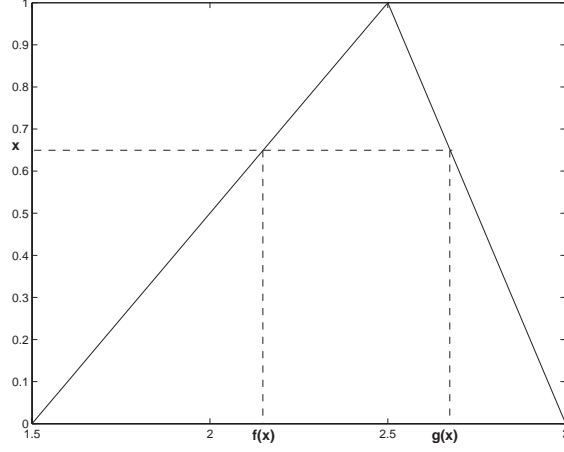


FIGURE 6. Equilibrium bidding function in Example 1.

Since  $\alpha'$  is a convex function, for the inequality above be attended it is enough to show

$$\alpha' \left( \frac{t + \varphi}{2} \right) \geq \frac{\alpha(\varphi) - \alpha(t)}{\varphi - t} = -\frac{2}{t + \varphi}$$

where the equality comes from the implicit definition of  $\varphi$ . However, this last equality is true because  $\alpha'(x) \geq -1/x$  for all  $x \in (0, 1]$ . ■

### Proof of the claims in Example 2.

First, let us show that there is no monotonic equilibria for this auction. By contradiction, assume that there is an increasing equilibrium bidding function. Then,  $P(t_i) = \frac{t_i - 1.5}{1.5}$  and condition (i)' is trivial. We have

$$\begin{aligned} \tilde{v}(x, y) &= (1.5x + 1.5) \left[ 1.5y + 1.5 - \frac{1.5x + 1.5}{2} \right] \\ &= \frac{9(x + 1)(2y - x + 1)}{8}. \end{aligned}$$

Thus, the necessary condition (ii) is not satisfied, because  $x > y$  implies

$$\int_y^x [\tilde{v}(x, \alpha) - \tilde{v}(\alpha, \alpha)] d\alpha = -\frac{3(x - y)^3}{8} < 0.$$

Thus, there is no monotonic equilibrium.

Now, we will show that there are multiple equilibria non-monotonic for this auction. Assume that there exists a bell-shaped equilibrium and that, for each  $x$ , there are two types,  $f(x)$  and  $g(x)$ , such that  $P(t_i) = x = \frac{3 - g(x) + f(x) - 1.5}{1.5}$ , which implies that  $g(x) = f(x) + 1.5(1 - x)$ . (See Figure 6).

Condition (i)' requires

$$\begin{aligned}
f(x) \left( \frac{f(y) + g(y)}{2} - \frac{f(x)}{2} \right) &= \frac{f(x) + g(x)}{2} \left( \frac{f(y) + g(y)}{2} \right) - \frac{f^2(x) + g^2(x)}{4} \\
&\Leftrightarrow \frac{f(y) + g(y)}{2} \left[ f(x) - \frac{f(x) + g(x)}{2} \right] = \frac{f(x)^2 - g(x)^2}{4} \\
&\Leftrightarrow \frac{f(y) + g(y)}{2} = \frac{f(x) + g(x)}{2}
\end{aligned}$$

Then,  $f(y) + g(y)$  is a constant, and we have  $f(x) = k + 3/4x$ . Since  $f(0) = 1.5$ ,  $k = 1.5$ . We obtain:

$$\begin{aligned}
\tilde{v}(x, y) &= \frac{f(x) + g(x)}{2} \left( \frac{f(y) + g(y)}{2} \right) - \frac{f^2(x) + g^2(x)}{4} \\
&= \left( \frac{9}{4} \right)^2 - \frac{(3/2 + 3/4x)^2 + (3 - 3/4x)^2}{4} \\
&= \left( \frac{9}{4} \right) \left[ 1 + \frac{x}{4} - \frac{x^2}{8} \right],
\end{aligned}$$

which satisfies condition (ii) because it is increasing in  $x$  on  $[0, 1]$ . Condition (iii) and (iv) are also satisfied, since

$$\tilde{b}(x) = \frac{1}{x} \int_0^x \tilde{v}(\alpha, \alpha) d\alpha = \frac{3(24 + 3x - x^2)}{32}$$

is increasing on  $[0, 1]$ . ■

### Proof for Example 3 - Spectrum Auction

Let us assume that  $t_i = (t_i^1, t_i^2, t_i^3)$  are independent and uniformly distributed on  $[\underline{s}^1, \bar{s}^1] \times [\underline{s}^2, \bar{s}^2] \times [\underline{s}^3, \bar{s}^3]$ , with  $\underline{s}^1, \underline{s}^2, \underline{s}^3 \geq 0$ . We have

$$v^1(t_i^1, t_i^2, t_i^3) = \frac{t_i^1}{N} - t_i^2 - \frac{N-1}{N} t_i^3 + \frac{N-1}{2N} \left[ (\bar{s}^1)^2 - (\underline{s}^1)^2 + (\bar{s}^3)^2 - (\underline{s}^3)^2 \right]$$

Let us denote by  $\bar{v}^1$  the expression in the first line above, that is,

$$\bar{v}^1(t_i^1, t_i^2, t_i^3) = \frac{t_i^1}{N} - t_i^2 - \frac{N-1}{N} t_i^3.$$

The conjugation  $P$  and the c.d.f.  $\tilde{P}$  are given by:

$$P(t_i^1, t_i^2, t_i^3) = [\Pr \{ (s^1, s^2, s^3) : \bar{v}^1(s^1, s^2, s^3) < \bar{v}^1(t_i^1, t_i^2, t_i^3) \}]^{N-1}.$$

and

$$\tilde{P}(k) = \left[ \Pr \left\{ (s^1, s^2, s^3) : \bar{v}^1(s^1, s^2, s^3) + \frac{N-1}{2N} \left[ (\bar{s}^1)^2 - (\underline{s}^1)^2 + (\bar{s}^3)^2 - (\underline{s}^3)^2 \right] < k \right\} \right]^{N-1}.$$

We can reparameterize the problem so that

$$\tilde{P}(k) = \left[ \Pr \left\{ (x, y, z) \in [0, 1]^3 : ax + by + cz < l(k) \right\} \right]^{N-1},$$

where  $a = (\bar{s}^1 - \underline{s}^1) / N > 0$ ,  $b = -(\bar{s}^2 - \underline{s}^2) < 0$ ,  $c = -(N-1)(\bar{s}^3 - \underline{s}^3) / N < 0$  and

$$l(k) = k - \frac{\underline{s}^1}{N} + \bar{s}^2 + \frac{N-1}{N}\underline{s}^3 - \frac{N-1}{2N} \left[ (\bar{s}^1)^2 - (\underline{s}^1)^2 + (\bar{s}^3)^2 - (\underline{s}^3)^2 \right].$$

It is elementary to obtain that, for a uniform distribution on  $[0, 1]^3$  and  $a > 0$ ,  $b < 0$ ,  $c < 0$  and  $k > b + c$ ,

$$\Pr \left\{ (x, y, z) \in [0, 1]^3 : ax + by + cz < l \right\} = \frac{(l - b - c)^3}{6abc}.$$

So,

$$\tilde{P}(k) = \frac{[l(k) - b - c]^{3(N-1)}}{(6abc)^{N-1}}$$

and

$$\begin{aligned} \tilde{v}(x, y) = & \left\{ \tilde{P}^{-1}(x) - \frac{N-1}{2N} \left[ (\bar{s}^1)^2 - (\underline{s}^1)^2 + (\bar{s}^3)^2 - (\underline{s}^3)^2 \right] \right\} y \\ & + E \left[ \frac{\sum_{j \neq i} (t_j^1 + t_j^3)}{N} \mid \max_{j \neq i} P(t_j) = y \right]. \end{aligned}$$

The candidate for the equilibrium on the first-price indirect auction is

$$\tilde{b}(x) = \frac{1}{x} \int_0^x \tilde{v}(\alpha, \alpha) d\alpha,$$

which is differentiable, with  $\tilde{b}'(x) = [\tilde{v}(x, x) - x] / x$ . Then, Theorem 3 tells us that there exists an equilibrium in regular pure strategies for this auction if and only if

$$\begin{aligned} \tilde{v}(x, x) - x = & \left\{ \tilde{P}^{-1}(x) - \frac{N-1}{2N} \left[ (\bar{s}^1)^2 - (\underline{s}^1)^2 + (\bar{s}^3)^2 - (\underline{s}^3)^2 \right] - 1 \right\} x \\ & + E \left[ \frac{\sum_{j \neq i} (t_j^1 + t_j^3)}{N} \mid \max_{j \neq i} v^1(t_j) = \tilde{P}^{-1}(x) \right] \geq 0. \end{aligned}$$

Depending on the values of  $\underline{s}^n$ ,  $\bar{s}^n$ , for  $n = 1, 2, 3$ , the above expression can be positive or negative. If it is always positive,  $\tilde{b}$  is increasing and it is the equilibrium of the indirect auction. In the other case, there is no equilibrium without ties. For instance, a sufficient condition for the existence of equilibrium in pure strategy is  $\frac{\underline{s}^1}{N} - \bar{s}^2 - \bar{s}^3 \frac{N-1}{N} - 1 \geq 0$ . ■

#### Proof for Example 4 - Job Market

We assume that there are two players with unidimensional signals uniformly distributed on  $[0, 1]$  and that  $m \in [0, 1]$ ,  $b \geq 0$ . Following the method given by Theorem 3, we first obtain

$$v^1(t_i) = am + \frac{c}{2} - b(t_i - m)^2.$$

We will consider two cases.

**First case:**  $m \leq 1/2$ . We have

$$P(t_i) = \begin{cases} 1 - 2m + 2t_i, & \text{if } 0 \leq t_i < m \\ 1 - 2t_i + 2m, & \text{if } m \leq t_i < 2m \\ 1 - t_i, & \text{if } 2m \leq t_i \leq 1. \end{cases}$$

So,

$$\tilde{v}(x, y) = \begin{cases} am + c(1 - y) - b(1 - x - m)^2, & \text{if } 0 \leq x, y < 1 - 2m \\ am + c(1 - y) - \frac{b}{4}(1 - x)^2, & \text{if } 0 \leq y < 1 - 2m \leq x \leq 1 \\ (a + c)m - b(1 - x - m)^2, & \text{if } 0 \leq x < 1 - 2m \leq y \leq 1 \\ (a + c)m - \frac{b}{4}(1 - x)^2, & \text{if } 1 - 2m \leq x, y \leq 1. \end{cases}$$

Now, it is easy to obtain, for  $x < 1 - 2m$ ,

$$\begin{aligned} \tilde{b}(x) &= \frac{1}{x} \int_0^x \left[ am + c(1 - y) - b(1 - y - m)^2 \right] dy, \\ &= am + c - b(1 - m)^2 - x \left[ \frac{c}{2} + b(m - 1) \right] - \frac{b}{3}x^2, \end{aligned}$$

which is increasing if  $c \leq \frac{2b(m+1)}{3}$ . For  $x > 1 - 2m$ ,

$$\begin{aligned} \tilde{b}(x) &= \frac{1}{x} \left\{ \frac{1}{6}(1 - 2m) [6am + 3c(1 + 2m) - 2b(1 - m + m^2)] \right. \\ &\quad \left. + \int_{1-2m}^x \left[ (a + c)m - \frac{b}{4}(1 - y)^2 \right] dy \right\} \\ \Rightarrow \tilde{b}(x) &= \frac{(1 - 2m)[2c - b(1 - 2m)]}{4x} + m(a + c) - \frac{b}{4} + \frac{b(3x - x^2)}{12} \end{aligned}$$

whose derivative can be simplified to

$$\tilde{b}'(x) = -\frac{(1 - 2m)[2c - b(1 - 2m)]}{4x^2} + \frac{b(3 - 2x)}{12}.$$

Since the term  $x^2(3 - 2x)$  is increasing, the bidding function will be increasing if and only if  $\tilde{b}'(1 - 2m) \geq 0$ , that is,

$$c \leq \frac{2b(1 - 2m)(1 + m)}{3}.$$

We conclude that in the case of  $m < 1/2$ , there exists a pure strategy equilibrium in regular strategies if and only if

$$c \leq \min \left\{ \frac{2b(m + 1)}{3}, \frac{2b(1 - 2m)(1 + m)}{3} \right\}.$$

**Second case:**  $m > 1/2$ . We have

$$P(t_i) = \begin{cases} t_i, & \text{if } 0 \leq t_i < 2m - 1, \\ 1 - 2m + 2t_i, & \text{if } 2m - 1 \leq t_i < m \\ 1 - 2t_i + 2m, & \text{if } m \leq t_i \leq 1 \end{cases}$$

and

$$\tilde{v}(x, y) = \begin{cases} am + cy - b(x - m)^2, & \text{if } 0 \leq x, y < 2m - 1 \\ am + cy - \frac{b}{4}(1 - x)^2, & \text{if } 0 \leq y < 2m - 1 \leq x \leq 1 \\ (a + c)m - b(x - m)^2, & \text{if } 0 \leq x < 2m - 1 \leq y \leq 1 \\ (a + c)m - \frac{b}{4}(1 - x)^2, & \text{if } 2m - 1 \leq x, y \leq 1. \end{cases}$$

For  $x < 2m - 1$ ,

$$\begin{aligned} \tilde{b}(x) &= \frac{1}{x} \int_0^x [am + cy - b(y - m)^2] dy, \\ &= am - bm^2 + x \left( \frac{c}{2} + bm \right) - \frac{b}{3}x^2, \end{aligned}$$

which is increasing in the considered interval if and only if  $c \geq \frac{2}{3}b(m - 2)$ .

For  $x > 2m - 1$ ,

$$\tilde{b}(x) = \frac{-2c(2m - 1) - b(2m - 1)^2}{4x} + \frac{12(a + c)m - b(3 - 3x + x^2)}{12}$$

which gives

$$\tilde{b}'(x) = \frac{2c(2m - 1) + b(2m - 1)^2}{4x^2} + \frac{b(3 - 2x)}{12}.$$

Following the same procedure of the first case,  $\tilde{b}'(x) \geq 0, \forall x \in [2m - 1, 1]$  if and only if

$$c \geq -\frac{2b(2m - 1)(1 + m)}{3}.$$

We conclude that, if  $m > 1/2$ , there exists a pure strategy equilibrium in regular strategies if and only if

$$c \geq \max \left\{ \frac{2}{3}b(m - 2), \frac{2b(1 - 2m)(1 + m)}{3} \right\}.$$

■

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